PHYS2063 Wave Physics
(Fall 2022)

Midterm Exam #3
Thursday 1 Dec 2022

There are **two questions** and you are to work both of them. You are welcome to use your textbook, notes, or any other resources, but you may not communicate with another human. Of course, if you have questions, you are encouraged to ask the person proctoring the exam.

Please start each problem on a new page in your exam booklet. Good luck!

(1) For quantum mechanical waves impinging on some "step", "barrier", or "well" in one dimension x, you saw some examples in class and homework where the transmission and reflection coefficients added to one. This makes sense, and is no accident, and in this problem I'm asking you to prove this *in general* no matter what the shape of the potential energy function. This is actually easier than you might think, and I will guide you through it.

(a) Write the continuity equation. Then reduce it to one dimension and where there is no time dependence. That is, assume "stationary state" solutions. (The result is very simple.)

(b) Integrate your result from **(a)** over the range $-\infty \leq x \leq +\infty$. Your answer will involve the current $j(x)$ evaluated at $\pm\infty$.

(c) Express $j(-\infty)$ in terms of incident and reflected currents, and $j(+\infty)$ in terms of the transmitted currents. (This is easy. Don't think too hard!)

(d) Use the result from **(c)** to show that $R + T = 1$, where R is the reflection coefficient and T is the transmission coefficient.

(2) This problem is to find the energy eigenstates of the infinite square potential well in *two dimensions* x and y. That is, the potential energy is zero for $0 \leq x \leq L$ and $0 \leq y \leq L$, and infinite elsewhere. The solution follows closely the way we worked the problem in one dimension, and makes use of some of the mathematics we used to solve the wave equation for standing waves, namely the technique of "separation of variables."

(a) Write the time-independent Schrödinger equation for a particle of mass m inside the well. Remember that the Laplacian operator in two dimensions is given by

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(b) Write the wave function $u(x,y) = X(x)Y(y)$ and then find separate ordinary differential equations for $X(x)$ and $Y(y)$ in terms of positive parameters k_x^2 and k_y^2 where the energy $E = (\hbar^2/2m)(k_x^2 + k_y^2)$.

(c) Use the boundary conditions to find the energy in terms of two independent positive integers n_x and n_y. What are the two lowest energies of the system, and how many states are there with each of these energies?

Solutions

(1) For time-independent wave functions in one dimension, the continuity equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = \frac{dj}{dx} = 0$$

Integrating this from $-\infty$ tp $+\infty$ just gives $j(+\infty) - j(-\infty) = 0$. Now $j(+\infty) = j_{\text{trans}}$, the transmitted flux. However, $j(-\infty) = j_{\text{inc}} - j_{\text{refl}}$. the difference between the incident and reflected flux. Therefore

$$j_{\text{trans}} - j_{\text{inc}} + j_{\text{refl}} = 0 \qquad \text{or} \qquad j_{\text{inc}} = j_{\text{trans}} + j_{\text{refl}}$$

which of course makes perfect sense. Dividing through by j_{inc} just gives $1 = T + R$.

(2) The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2 u(x,y) + V(x,y)u(x,y) = -\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial y^2} = Eu(x,y) = \frac{\hbar^2}{2m}(k_x^2 + k_y^2)u(x,y)$$

Writing $u(x,y) = X(x)Y(y)$ and then dividing through by $u(x,y)$ gives

$$\left[\frac{1}{X}\frac{d^2X}{dx^2} + k_x^2\right] + \left[\frac{1}{Y}\frac{d^2Y}{dy^2} + k_y^2\right] = 0$$

The first term in brackets only depends on x, and the second only depends on y, so they both must equal the same constant but with opposite signs. We might as well make the constant zero because otherwise only means redefining k_x and k_y. Therefore we have our old friend differential equations for simple oscillations:

$$\frac{d^2X}{dx^2} = -k_x^2 X(x) \qquad \text{and} \qquad \frac{d^2Y}{dy^2} = -k_y^2 Y(y)$$

The solutions are just sines and cosines. The boundary conditions require that $u(x,y) = 0$ at $x = 0$ and $y = 0$, so there is only the sine solutions. The boundary conditions also require that $u(x,y) = 0$ at $x = L$ and $y = L$ and setting the sines equal to zero there imply that $k_x = n_x\pi$ and $k_y = n_y\pi$ where n_x and n_y are positive integers. Therefore

$$E = \frac{\hbar^2}{2m}(k_x^2 + k_y^2) = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2)$$

The lowest energy is for $n_x = n_y = 1$, that is, $E_{1,1} = \hbar^2\pi^2/mL^2$. The next highest energy is

$$E_{2,1} = \frac{\hbar^2\pi^2}{2mL^2}(2^2 + 1^2) = 5\frac{\hbar^2\pi^2}{2mL^2} = E_{1,2}$$

That is, there are two states with the same "first excited state energy." We say that the first excited state is "doubly degenerate."