PHYS2063 Wave Physics
(Fall 2022)

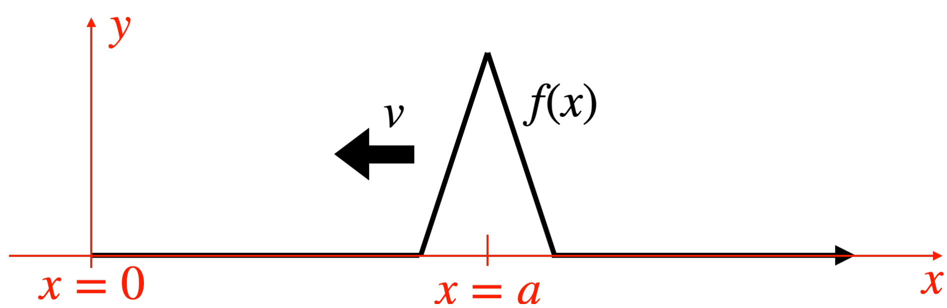
Midterm Exam #2
Thursday 27 Oct 2022

There are **three questions** and you are to work both of them. You are welcome to use your textbook, notes, or any other resources, but you may not communicate with another human. Of course, if you have questions, you are encouraged to ask the person proctoring the exam.

Please start each problem on a new page in your exam booklet.

Good luck!

(1) A semi-infinite string lies along the x-axis with one end at $x = 0$. The end of the string at $x = 0$ is *not fixed* but rather floats freely but always keeps *zero slope* at $x = 0$. A pulse with the shape $f(x)$ at time $t = 0$ as shown is moving to the left with speed v towards the floating end.



The function $f(x)$ is localized with a peak at $x = a$, and is much narrower than a. Find an equation for the vertical motion of the string in terms of $f(x)$, and describe this motion in words. Be as precise as you can with time in terms of a/v.

(2) An electric field in vacuum permeates all space and has the form

$$\vec{E}(x,y,z) = E_0\left[\hat{i}\left(\frac{x}{a}\right)^2 + \hat{j}\frac{z}{a} + \hat{k}\frac{y}{a}\right]$$

Find the amount of charge contained in a cube of side length a lying in the first octant (x, y, and z all positive) with one corner at the origin.

(3) The magnetic field of an electromagnetic wave propagating in vacuum is given by

$$\vec{B}(\vec{r},t) = \hat{\epsilon}B_0\cos(\vec{k}\cdot\vec{r} - \omega t)$$

where $\omega = |\vec{k}|c$. If the wave is propagating in the xy plane along the line $x = y$ in the direction of increasing x and increasing y, and the magnetic field vector also lies in the xy plane, find $\vec{k}$, $\hat{\epsilon}$, and the electric field $\vec{E}(\vec{r},t)$. Also show that $\vec{E}\times\vec{B}$ is in the expected direction.

Solutions

(1) The equation of the string as shown is $f(x+vt)$, since it is moving towards the left. We can add to this any functional form moving to the right, that is, a function of $x-vt$. If we choose

$$u(x,t) = f(x+vt) + f(-x+vt) \qquad \text{then} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = f(vt) - f(vt) = 0$$

and the boundary condition is satisfied. However $f(-x+vt)$ does not exist at $x>0$ until $t>a/v$, so it is in the unphysical region where the string does not exist until the pulse makes contact with the boundary at $x=0$. Therefore, the pulse moves to the left where it adds to the pulse coming to the right, with a peak value at $x=0$ of $2f(a)$ for $t=a/v$, after which it is the rightward moving pulse that is in the physical region.

(2) This is just an application of Gauss' Law.

$$Q = \int_V \rho\, dV = \frac{1}{4\pi}\int_V \vec{\nabla}\cdot\vec{E}\, dV = \frac{1}{4\pi}\int_V E_0\frac{2x}{a^2}\, dV = \frac{E_0}{4\pi a^2}a^2\int_0^a 2x\, dx = \frac{E_0 a^2}{4\pi}$$

(3) We have $\vec{k} = k(\hat{\imath}+\hat{\jmath})/\sqrt{2}$ and $\hat{\epsilon} = \hat{\imath}a + \hat{\jmath}b$ with $\vec{k}\cdot\hat{\epsilon} = 0$ so $a+b=0$ and $\hat{\epsilon} = (\hat{\imath}-\hat{\jmath})/\sqrt{2}$ since $a^2+b^2=1$. (Or we could have switched the signs of a and b.) So we have

$$\vec{B}(\vec{r},t) = B_0\cos\left(\frac{1}{\sqrt{2}}kx + \frac{1}{\sqrt{2}}ky - \omega t\right)\left[\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}\right]$$

We get the electric field from Ampere's Law as modified by Maxwell, namely

$$\begin{aligned}
\frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{B} &= -B_0\frac{k}{\sqrt{2}}\sin\left(\frac{1}{\sqrt{2}}kx + \frac{1}{\sqrt{2}}ky - \omega t\right)\hat{k}\frac{1}{\sqrt{2}}\left[(-1)-(1)\right] \\
&= kB_0\hat{k}\,\sin\left(\frac{1}{\sqrt{2}}kx + \frac{1}{\sqrt{2}}ky - \omega t\right) = kB_0\hat{k}\,\sin\left(\vec{k}\cdot\vec{r}-\omega t\right) \\
\text{so} \qquad \frac{\partial\vec{E}}{\partial t} &= \omega B_0\hat{k}\,\sin\left(\vec{k}\cdot\vec{r}-\omega t\right) \\
\text{and} \qquad \vec{E}(\vec{r},t) &= -B_0\hat{k}\,\sin\left(\vec{k}\cdot\vec{r}-\omega t\right)
\end{aligned}$$

Note that the electric field has the same magnitude as the magnetic field, which it must, and is polarized in the negative z-direction, perpendicular to the polarization of the magnetic field. The direction of the Poynting vector is

$$\hat{k}\times\left[\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}\right] = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$$

which is the direction of $\vec{k}$, which is correct.