PHYS2063 Wave Physics
(Fall 2022)

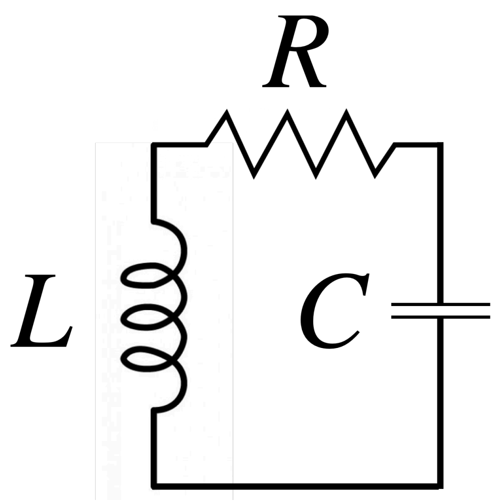
Midterm Exam #1
Thursday 29 Sep 2022

There are **two questions** and you are to work both of them. You are welcome to use your textbook, notes, or any other resources, but you may not communicate with another human. Of course, if you have questions, you are encouraged to ask the person proctoring the exam.

Please start each problem on a new page in your exam booklet.

Good luck!

(1) A capacitor C, inductor L, and resistor R are connected in series as in the diagram here:



The initial charge on the capacitor is zero, and the initial current through the resistor is i_0. Find the current $i(t)$ as a function of time, assuming that R is less than $2\sqrt{L/C}$.

(2) A string with linear mass density μ is held horizontal and taught with tension T with its ends fixed to the x-axis at $x = 0$ and $x = L$. The vertical position of the string is given by $y = u(x, t)$ at position x and time t. The string is horizontal initially, but has an initial vertical velocity profile given by

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 2v \sin\left(\frac{2\pi x}{L}\right)$$

where $v = \sqrt{T/\mu}$. Find the time dependent shape of the string, that is, determine the function $u(x, t)$.

Solutions

(1) The differential equation for the charge $q(t)$ on the capacitor is

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0 \qquad \text{or} \qquad \ddot{q} + 2\beta\dot{q} + \omega_0^2 q = 0$$

where $\beta = R/2L$ and $\omega_0^2 = 1/LC$. Subsituting $q(t) = Ae^{\alpha t}$ gives

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0 \qquad \text{so} \qquad \alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2} = -\beta \pm i\omega_1$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \sqrt{1/LC - R^2/4L^2}$ which is a real number because $1/LC - R^2/4L^2 = (1/4L^2)(4L/C - R^2) > 0$. Therefore, the general solution is $q(t) = e^{-\beta t}(A\cos\omega_1 t + B\sin\omega_1 t)$. Now $q(0) = A = 0$, so $q(t) = Be^{-\beta t}\sin\omega_1 t$. Now $i(t) = \dot{q} = \omega_1 Be^{-\beta t}\cos\omega_1 t - \beta Be^{-\beta t}\sin\omega_1 t$ which gives $i_0 = i(0) = \omega_1 B$ so $B = i_0/\omega_1$ and the complete solution is

$$q(t) = \frac{i_0}{\omega_1}e^{-\beta t}\sin\omega_1 t \qquad \text{and} \qquad i(t) = \dot{q}(t) = i_0 e^{-\beta t}\cos\omega_1 t - \frac{\beta}{\omega_1}i_0 e^{-\beta t}\sin\omega_1 t$$

(2) The general solution to the wave equation, from the method of separation of variables, subject to boundary conditions $u(0,t) = u(L,t) = 0$ is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi vt}{L}\right) + B_n \sin\left(\frac{n\pi vt}{L}\right)\right]$$

Since "the string is horizontal initially", one initial condition is $u(x,0) = 0$, which implies that $A_n = 0$ for all n. We then have

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi vt}{L}\right) \qquad \text{so} \qquad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi v}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right)$$

Therefore, the second initial condition becomes

$$\sum_{n=1}^{\infty} B_n \left(\frac{n\pi v}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = 2v\sin\left(\frac{2\pi x}{L}\right)$$

It is easy to see that $B_n = 0$ for all n except $n = 2$ in which case $B_2(n\pi v/L) = 2v$, and,

$$u(x,t) = \frac{L}{\pi}\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{2\pi vt}{L}\right)$$