Find solution to the ordinary differential equation

$$x'(t) = \frac{dx}{dt} = f(t,x)$$

$$x(0) = x_0$$ (initial condition)

The ODE gives the rate of change of $x(t)$ in time or slope of the tangent line at each point in time.

The solution to an ODE is some function $x(t)$.
When we can write down a solution for all times that's the analytical solution.

Usually it's challenging to determine an explicit formula for $x(t)$ so we numerically approximate it at discrete points

Numerical approximation of x

Knowing the value of x at some starting point t_0 ie. x_0
Find an approximation of x at discrete times

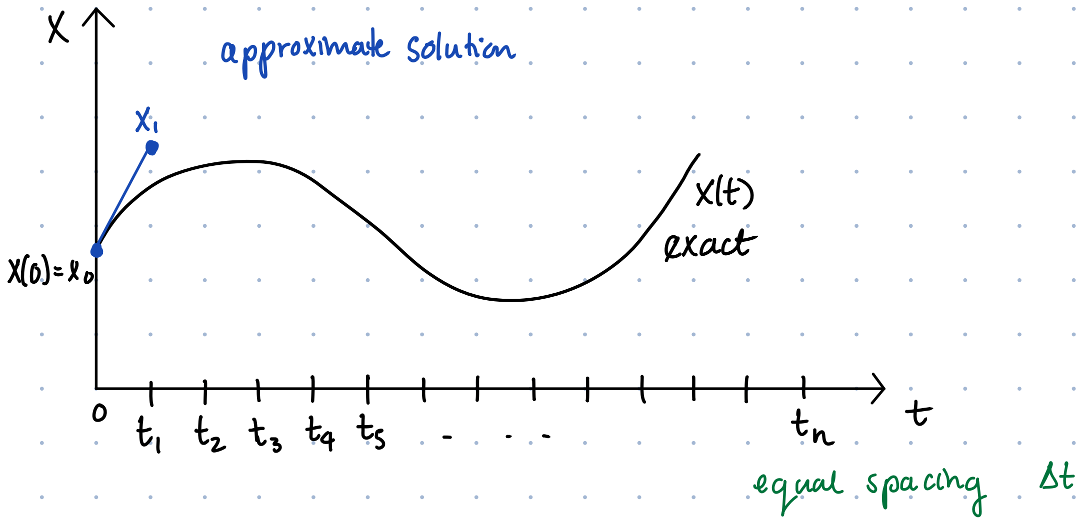
$$t_1, t_2, t_3, \ldots, t_n \quad \text{where} \quad t_j - t_{j-1} = \Delta t$$
$$\text{for all} \quad j = 1, \ldots, n$$

Call the numerical approximation at time t_j

$$x_j \approx x(t_j) = x(j\Delta t)$$

Euler's Method

Euler's Method uses the slope of x at each time step to extrapolate the solution to the next time step.



Slope to tangent line to the graph of $x(t)$ at t_0 is $f(t_0, x_0)$

Now use this slope to estimate the slope between x_0 and $x(t_1)$

$$\frac{x(t_1) - x_0}{\Delta t} \approx f(t_0, x_0)$$

$$x(t_1) \approx x_0 + \Delta t * f(t_0, x_0) = x_1$$

approximates $x(t_1)$

$$x_1 = x_0 + \Delta t * f(t_0, x_0)$$

Now assume x_1 is the exact solution, the slope to tangent line at x_1 is $f(t_1, x_1)$ then

$$x_2 = x_1 + \Delta t * f(t_1, x_1)$$

In general
$$\boxed{x_j = x_{j-1} + \Delta t * f(t_j, x_j)}$$

What is the error made in one step of Euler method?

Suppose $x(t_{j-1}) = x_{j-1}$

approx $\quad x_j = x_{j-1} + \Delta t \cdot f(x_{j-1})$

true $\quad x(t_j) = x(t_{j-1} + \Delta t)$

$$= x(t_{j-1}) + \Delta t * x'(t_{j-1}) + \frac{1}{2}(\Delta t)^2 x''(t_{j-1}) + O(\Delta t^3)$$

$$e_j = x(t_j) - x_j = \frac{1}{2}(\Delta t)^2 x''(t_{j-1}) + \underbrace{O(\Delta t^3)}$$

error at step j

higher order terms become insignificant as Δt becomes smaller

There are $n = \frac{t_f}{\Delta t}$ steps $\Rightarrow O\left(\frac{1}{\Delta t}\right)$ steps

Per step accumulate error $O(\Delta t^2)$

Global error $\rightarrow O\left(\frac{1}{\Delta t}\right) O(\Delta t^2) = O(\Delta t)$

Euler Method is a first order method

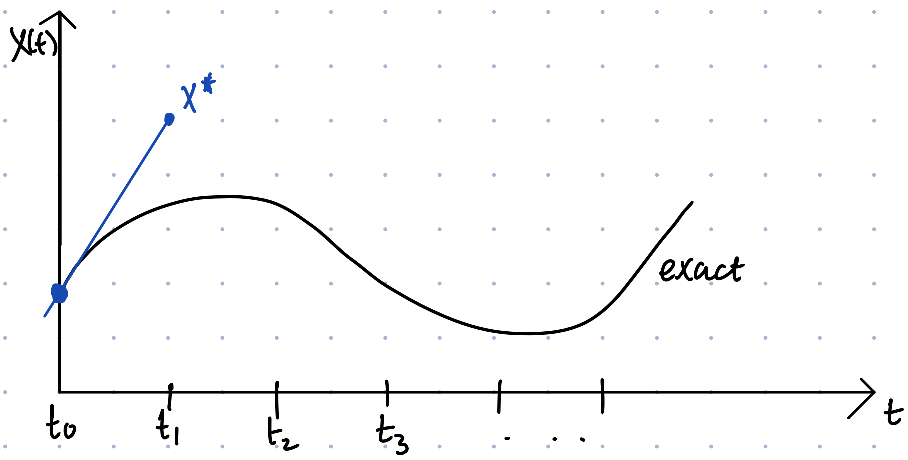
Reduce time step size Δt by 2, error goes down a factor of 2 i.e. a linear decrease in error.

We can do better i.e. a quadratic decrease in error or more.

Runge-Kutta Methods

Runge-Kutta methods are a class of numerical methods for approximating ODEs that use information on the 'slope' at more than one point to extrapolate the solution to a future time step.

eg. Explicit Trapezoidal Rule / Heun's Method.



Use slope information at left at right endpoint of time step interval

Starting with interval $[t_0, t_1]$

Slope of tangent at left $= f(t_0, x_0)$

We don't know the value of x at t_1 yet

Approximate with Euler's Method $x^* = x_0 + \Delta t * f(t_0, x_0)$

Now find slope of tangent at right point

$= f(t_1, x_0 + \Delta t * f(t_0, x_0))$

Take an average of the two slopes and use this average to find x_1

$$\text{Average} = \frac{f(t_0, x_0) + f(t_1, x_0 + \Delta t * f(t_0, x_0))}{2}$$

$$x_1 = x_0 + \Delta t * \text{Average}$$

$$= x_0 + \Delta t * \left(\frac{f(t_0, x_0) + f(t_1, x_0 + \Delta t * f(t_0, x_0))}{2}\right)$$

In general to find x_j given x_{j-1}

$$x_j = x_{j-1} + \frac{\Delta t}{2}(k_1 + k_2)$$

where $k_1 = f(t_{j-1}, x_{j-1})$

$k_2 = f(t_j, x_{j-1} + \Delta t * k_1)$

This is a second order method $O((\Delta t)^2)$

so when the step size is cut in half the error goes down by a factor of 4.

Runge-Kutta 4 (RK4)

Arguably the most popular Runge-Kutta method

RK4 because it is fourth order $O(\Delta t^4)$

if $\Delta t \to \frac{\Delta t}{2}$, error goes down by factor of 8

RK4 formula

$$x_{j+1} = x_j + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(t_j, x_j)$$

$$k_2 = f(t_j + \frac{\Delta t}{2}, x_j + \frac{\Delta t}{2} k_1)$$

$$k_3 = f(t_j + \frac{\Delta t}{2}, x_j + \frac{\Delta t}{2} k_2)$$

$$k_4 = f(t_{j+1}, x_j + \Delta t * k_3)$$

where k_1, k_2, k_3, k_4 are slopes at points between t_j and $t_{j+1} = t_j + \Delta t$