Phase Portraits

1 Phase Portraits

Consider an autonomous system of ordinary differential equations (ODEs)

$$x'(t) = f(x, y)$$
$$y'(t) = g(x, y).$$

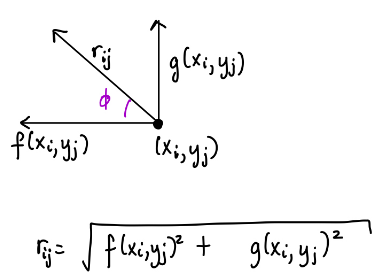
Here autonomous refers to the fact that the functions on the right hand side of the equation do not explicitly depend on time.

- The phase portrait is a graphical tool to visualize how the solutions of a given system of differential equations behaves in the long run. It is a representative set of solutions for the system of ODEs, plotted as parametric curves (with t as the parameter) on the Cartesian plane tracing the path of each particular solution.
- The plane where the phase portrait lives is called the **phase plane**.
- The parametric curves traced by the solutions are called their **trajectories**.

Analogy for a parametric curve: To help visualize what a parametric curve is: suppose we have a stream of water and we drop a stick in it, the point $\big(x(t), y(t)\big)$ represents the location of the stick in the stream at a time t and the parametric curve will be a trace of all the locations of the stick as it moves along the stream.

1.1 Drawing a phase portrait

- Define a grid in the phase plane with points denoted (x_i, y_j) for some indices i, j.
- For each grid point plug (x_i, y_j) into scalar functions f and g to find x' and y' at that point. (x', y') represents the tangent vector of the solution's trajectory. It gives the instantaneous direction of motion at that point.
- The magnitude of the vector at (x_i, y_j) on the grid is given by $r_{ij} = \sqrt{f(x_i, y_j)^2 + g(x_i, y_j)^2}$, while the direction is given by $\arctan\left(\dfrac{g(x_i, y_j)}{f(x_i, y_j)}\right)$.
- The magnitudes of the vectors can be normalized so that each vector has length 1.



1.2 Fixed points and classification

Fixed points also known as equilibrium points, critical points or stationary points are points (x, y) where $x' = 0$ and $y' = 0$. These points are essential in determining the long-term behavior of solutions. They can be classified by the type/shapes formed by the trajectories about each fixed point and by stability.

1.3 Linear Systems

When the system of ODEs is linear we can use a matrix to write $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{x} = (x, y)$. In the case that $\mathbf{A}$ is nonsingular, that means $\mathbf{x}' = \mathbf{A}\mathbf{x} = 0$ only has a single fixed point at $(0, 0)$. To classify this fixed point, the eigenvalues of $\mathbf{A}$ have to be identified i.e. λ_j such that $\mathbf{A}\mathbf{x_j} = \lambda_j \mathbf{x_j}$ where $\mathbf{x}_j$ are the eigenvectors.

1.4 Nonlinear Systems

When the system of ODEs is nonlinear we cannot use a matrix to denote the right handside. We can still find fixed points by setting $x' = 0$ and $y' = 0$. A nonliner system can have zero, one, two, three, or any number of fixed points. Since there are many fixed points on the phase portrait, each trajectory could be influenced by more than one fixed point. The type and stability of each fixed point needs to be determined locally, in a small neighborhood of the fixed point in the phase plane. We can approximate the behavior of trajectories near the fixed point by using a linearization of f and g about the fixed point. We compute the Jacobian matrix consisting of all first-order partial derivatives of the system and evaluate it at each fixed point to find

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f(x_i, y_j)}{\partial x} & \frac{\partial f(x_i, y_j)}{\partial y} \\ \frac{\partial g(x_i, y_j)}{\partial x} & \frac{\partial g(x_i, y_j)}{\partial y} \end{bmatrix} \tag{1}$$

where (x_i, y_j) denotes an arbitrary fixed points. We use eigenvalues of $\mathbf{J}$ to determine the type and stability of the fixed point.

1.5 Classifications of fixed points

1.5.1 Terms:

- Asymptotically stable - all trajectories of the solutions converge to the fixed point as $t \to \infty$.
- Unstable - all(save for a few in case of saddle point) trajectories move away from the fixed point as $t \to \infty$.
- Stable(neutrally stable) - trajectories stay within a fixed orbit that's within a finite distance away from the fixed point

1.5.2 Classification by eigenvalue type

1. λ_1 and λ_2 are real, distinct, and positive
   The fixed point is a **node**. Asymptotically stable if eigenvalues are negative, otherwise unstable.
   At a node, trajectories are tangential to the slow eigendirection - the eigendirection whose λ has the smallest magnitude.

2. λ_1 and λ_2 are real and have opposite signs
   The fixed point is a **saddle point**. It is always unstable.
   The stable manifold is the eigendirection with the negative eigenvalue.

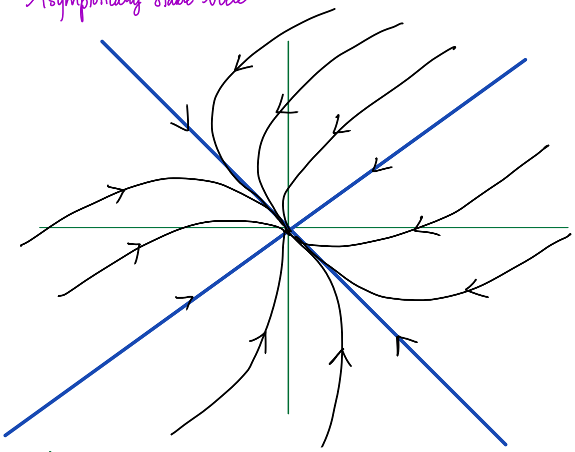
3. $\lambda_1 = \lambda_2$ real with two linearly independent eigenvectors
   The fixed point is a **star/proper node**. Asymptotically stable if eigenvalue is negative, otherwise unstable.

4. $\lambda_1 = \lambda_2$ real with only one eigenvector
   The fixed point is an **degenerate/improper node**. Asymptotically stable if eigenvalue is negative, otherwise unstable.

5. λ_1 and λ_2 are complex conjugate pairs with zero real part
   The fixed point is a **center**. It is neutrally stable.

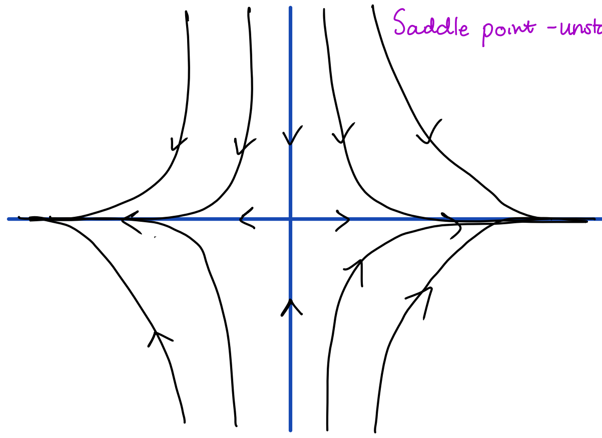
6. λ_1 and λ_2 are complex conjugate pairs with nonzero real part
   The fixed point is a **spiral**. Trajectories spiral away (unstable) from fixed point if $\mathrm{Re}(\lambda) > 0$ and spiral towards (asymptotically stable) the fixed point if $\mathrm{Re}(\lambda) < 0$.

Asymptotically Stable Node

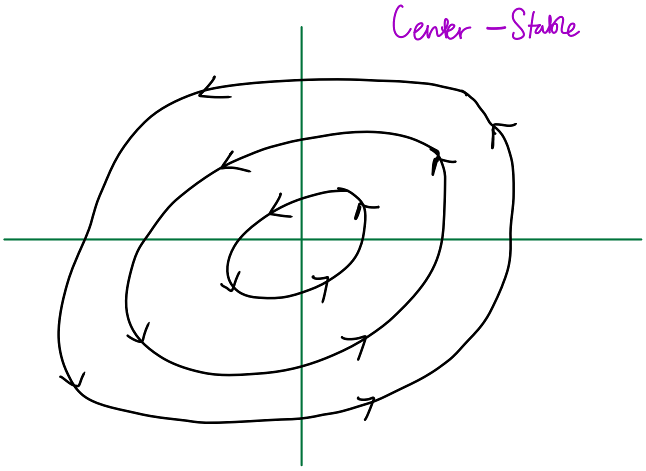


Arrows reversed for unstable

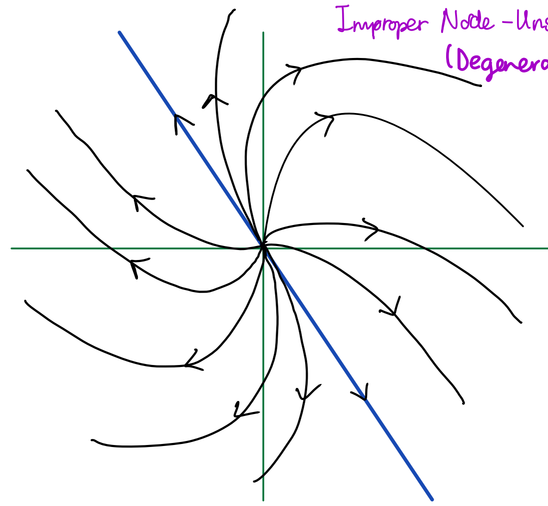
Saddle point - unstable



Center - Stable

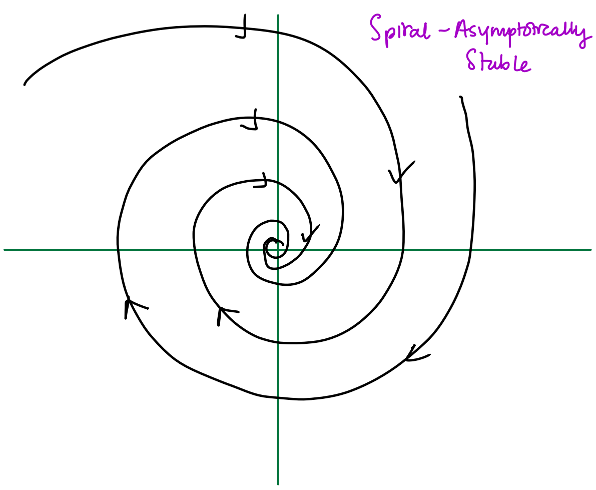


Improper Node - Unstable (Degenerate)



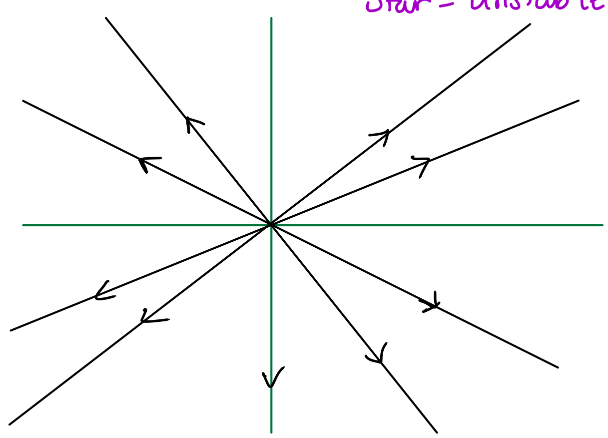
Arrows reversed for stable

Spiral - Asymptotically Stable



Arrows reversed for unstable

Star - Unstable



Arrows reversed for stable

Example

Find and classify fixed points for the Lotka-Volterra predator-prey model

$$x' = x - xy$$
$$y' = xy - y.$$

Fixed points: Solve $x' = 0$ and $y' = 0$ simultaneously to find the following points $(0, 0)$ and $(1, 1)$.

The Jacobian is

$$\begin{bmatrix} 1-y & -x \\ y & x-1 \end{bmatrix} \tag{2}$$

Evaluating at $(0, 0)$ results in

$$\mathbf{J} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

There are two real, distinct eigenvalues of opposite signs $\lambda_1 = 1, \lambda_2 = -1$ so the fixed point is an unstable saddle point. Does it make sense to have a saddle point here?

Evaluating at $(1, 1)$ results in

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues can be obtained by finding the determinant of $J - \lambda I$ and setting it to zero:

$$\det\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}\right) = \lambda^2 + 1 = 0$$

So the eigenvalues are $\lambda_1 = i, \lambda_2 = -i$ and the fixed point is a center and hence stable. Centers are closed trajectories signifying periodic solutions. Population cycles of growth and decline emerge.

2 In-class exercises

1. Find the fixed points of all the models in the code `temple_abm_population_local.m`. You can use Matlab's `solve` function.
2. Find the Jacobian function for each of the models.
3. Classify the fixed points you find by plugging in the point into the Jacobian function and using Matlab's `eigs` function
4. Run the code for each model and anaylze the phase portrait. Do your classifications match up with the phase portrait?
5. What does the phase portrait tell us about solutions of the system of equations in each model?
6. On the phase portrait, the trajectory/solution matching up with initial condition given is traced, try a different initial condition or different model parameters to see how the phase portraits and trajectories change.

The trajectory is computed using a numerical method called Euler's method which we will discuss in the coming lectures.