"The one about homogenization"

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The theory of homogenization deals with finding the effective behavior of systems whose coefficients oscillate on the microscale. Application is found in the study of composite or crystaline materials where the interest is in finding the consequence of a given microstructure or in achieveing materials possessing physical properties which aren't possessed by its constitutive materials.

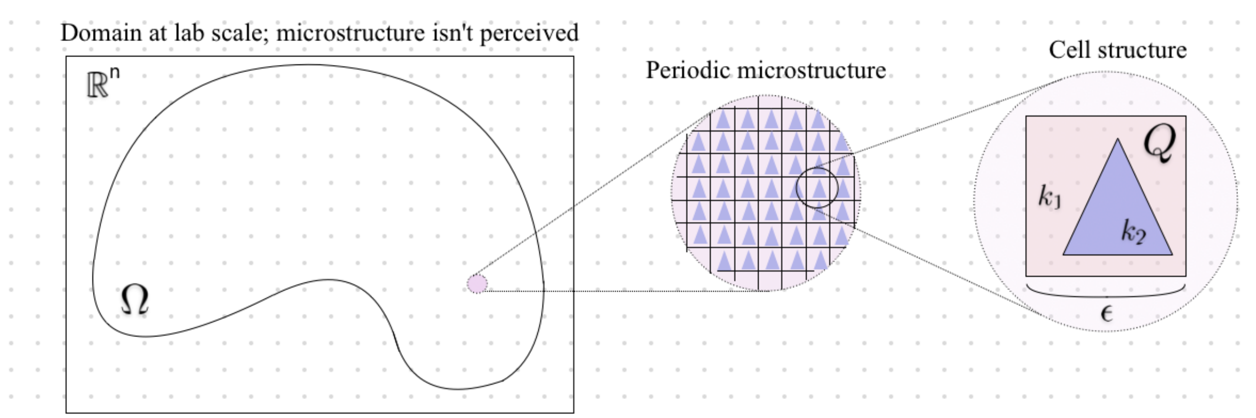
In these notes we discuss the homogenization of elliptic PDEs of the form,

$$\begin{cases} \nabla \cdot A(z)\nabla u(z) = f(z), & z \in \Omega \\ u(z) = g(z), & z \in \partial\Omega, \end{cases} \tag{0.1}$$

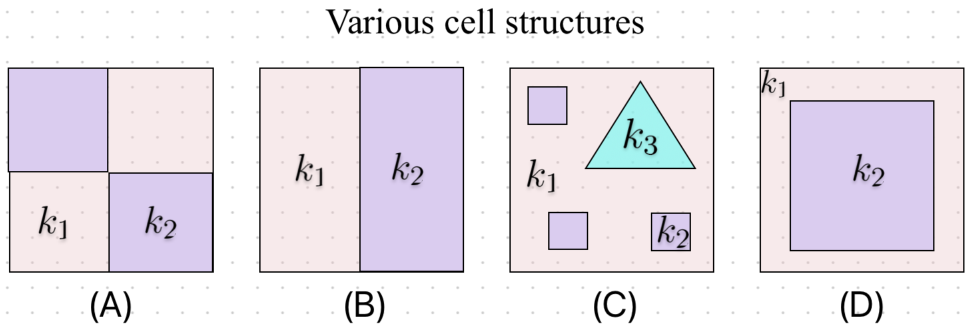
where there exists $\beta > \alpha > 0$ such that for all $z \in \Omega$ and $\xi \in \mathbb{R}^n$, $A(z)$ is symmetric,

$$\alpha_0|\xi|^2 \leq \xi^T A(z)\xi \leq \beta|\xi|^2, \tag{0.2}$$

and the coefficients are $[0, \epsilon]^d$-periodic (See Figure 1). Such equations describe the steady state (heat/electrical) conductivity with source, f. In Figure 2, an example of a domain on which the conducivity, A, oscillates between constant values $A = k_1$ and $A = k_2$ on the micro-scale.



To highlight the effect of period microstructure in such systems, in figure I give a list of possible cell structures consisting of two materials of different conductivity with various volume fractions:



Consider materials resulting from microstructures given by (A) and (B). If k_1 and k_2 are isotropic, one might expect material (A) to retain isotropic behavior and (B) to act anisotropic. In fact, letting $k_1 = cI_n$ for some $c > 0$ and $k_2 \equiv 0$, material (B) would conduct only in the e_2 direction and not at all in the e_1 direction.

Generally, two types of results are sought ([1]):

1. An effective conductivity tensor, A^0, which does not depend on ϵ, where A^0 is related to problem (0.1) in the following way: Given $\{u^\epsilon\}$ where u^ϵ solves (0.1) weakly in $H^1(\Omega)$, A^0 is the effective conductivity tensor if u^ϵ converges weakly to u^0 in $H^1(\Omega)$ and $A(\frac{x}{\epsilon})\nabla u^\epsilon$ converges weakly to $A^0 u^0$ where u^0 solves

$$\begin{cases} \nabla \cdot A^0(z)\nabla u^0(z) = f(z), & z \in \Omega \\ u^0(z) = g(z), & z \in \partial\Omega. \end{cases} \tag{0.3}$$

2. An approximation, ϕ^ϵ, of u^ϵ with explicit convergence bounds in $H^1(\Omega)$; i.e. something of the form

$$\|\phi^\epsilon - u^\epsilon\|_{H^1(\Omega)} \leq c\epsilon^\alpha.$$

   for some explicitly given power $\alpha > 0$. To make this useful, it should also be required that ϕ^ϵ has a simple construction (much simpler than that of u^ϵ).

Multi-scale expansion

In this section, I give the standard formal derivation of the formulas for the effective conductivity, A^0, as well as the approximation ϕ^ϵ (as found in [2] and [3]). First, so that we don't need to redefine A for each ϵ as $\epsilon \to^+ 0$, we may consider A to be $[0,1]^d$-periodic and u satisfying the equation

$$\begin{cases} \nabla \cdot A\left(\frac{z}{\epsilon}\right) \nabla u^\epsilon(z) = f(z), & z \in \Omega \\ u^\epsilon(z) = g(z), & z \in \partial\Omega. \end{cases} \tag{0.4}$$

This is equivalent to the earlier formulation. To guide our study of the problem, we note that there are two distinct scales: the lab scale which remains $O(\text{diam}(\Omega))$, and the micro scale, ϵ. The we introduce two new variables to account for these:

$$x := z, \qquad \text{and} \qquad y := \frac{z}{\epsilon}.$$

Variations in x capture variations in z: if $\Delta x = \Delta z$; while variations in y capture imperceivable, micro-scale variations in z: if $\Delta y \sim O(1)$ then $\Delta z \sim O(\epsilon)$. To study the effective behavior of the system as $\epsilon \to^+ 0$, it is advantageous to use a formal two-scale expansion of u^ϵ:

$$u^\epsilon(z) = u^0(x,y) + \epsilon u^1(x,y) + \epsilon^2 u^2(x,y) + \dots. \tag{0.5}$$

To use this expansion, we require $u^i(x,y)$ to be periodic in y for $i = 0, 1, \dots$, and we use the differentiation rule:

$$\nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y.$$

We begin applying this to (0.1):

$$\begin{aligned}\left(\nabla_x + \frac{1}{\epsilon}\nabla_y\right)\left(u^0(x,y) + \epsilon u^1(x,y) + \epsilon^2 u^2(x,y) + \dots\right) &= \\ &= \frac{1}{\epsilon}\left(\nabla_y u^0\right) + \left(\nabla_x u^0 + \nabla_y u^1\right) + \epsilon\left(\nabla_x u^1 + \nabla_y u^2\right) + O(\epsilon^3).\end{aligned}$$

so that (0.1) becomes

$$\begin{aligned}\left(\nabla_x + \frac{1}{\epsilon}\nabla_y\right)\cdot A(y)\left(\nabla_x + \frac{1}{\epsilon}\nabla_y\right)&\left(u^0(x,y) + \epsilon u^1(x,y) + \epsilon^2 u^2(x,y) + \dots\right) = \\ &= \frac{1}{\epsilon^2}\nabla_y \cdot A(y)\nabla_y u^0 + \frac{1}{\epsilon}\left(\nabla_x \cdot A(y)\nabla_y u^0 + \nabla_y \cdot A(y)\left(\nabla_x u^0 + \nabla_y u^1\right)\right) \\ &\quad + \nabla_x \cdot A(y)\left(\nabla_x u^0 + \nabla_y u^1\right) + \nabla_y \cdot A(y)\left(\nabla_x u^1 + \nabla_y u^2\right) + \dots \\ &= f(x).\end{aligned}$$

for $z \in \Omega$ and $y \in Q$, with boundary conditions $u^0(x,y) = f(x)$ for $x \in \partial\Omega$, $u^i(x,y) = 0$ for $x \in \partial\Omega$ for $i > 1$, and with periodic boundary conditions for $u^i(x,y)$ for each i with respect to y. Equating terms of the same order, we get a system of three equations:

$$\begin{cases}\nabla_y \cdot A(y)\nabla_y u^0(x,y) & = 0, \\ \nabla_x \cdot A(y)\nabla_y u^0 + \nabla_y \cdot A(y)\left(\nabla_x u^0 + \nabla_y u^1\right) & = 0, \\ \nabla_x \cdot A(y)\left(\nabla_x u^0 + \nabla_y u^1\right) + \nabla_y \cdot A(y)\left(\nabla_x u^1 + \nabla_y u^2\right) & = f(x)\end{cases} \tag{0.6}$$

Consider the first equation,

$$\nabla_y \cdot A(y)\nabla_y u^0(x,y) = 0.$$

Viewing this as an homogeneous elliptic PDE in y with periodic boundary conditions, we seek solutions in the class, $H^1(Q_p)$(Q-periodic H^1 functions). Since this is homogeneous with periodic boundary conditions, for each fixed x, $u^0(x,y)$ must be constant in y. Thus,

$$u^0(x,y) = u^0(x).$$

Now consider the second equation, which, due to the observation that u^0 is constant in y, now simplifies to

$$\nabla_y \cdot A(y)\left(\nabla_x u^0(x) + \nabla_y u^1\right) = 0.$$

Consider the solutions, $v^j(y)$ (unique up to additive constant) of the following problems

$$\nabla_y \cdot A(y)\left(e_j + \nabla_y v^j\right) = 0,\ j = 1, \ldots, n$$

Where $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$. By Linearity,

$$\nabla_y u^1(x,y) = \nabla_y \left(v(y)^T \nabla_x u^0(x)\right) \quad \text{where} \quad v = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

For the last equation,

$$\nabla_x \cdot A(y)\left(\nabla_x u^0 + \nabla_y u^1\right) + \nabla_y \cdot A(y)\left(\nabla_x u^1 + \nabla_y u^2\right) = f(x),$$

Assume for now that all quantities are smooth. Then applying the averaging operator (over the variable, y), $\langle\cdot\rangle$, the above equation simplifies to

$$\langle \nabla_x \cdot A(y)\left(\nabla_x u^0 + \nabla_y u^1\right) - f\rangle = 0.$$

Given that f is a function of only x, we have,

$$\nabla_x \cdot \langle A(y)\left(\nabla_x u^0 + \nabla_y u^1\right)\rangle = f.$$

By the definition of $v(y)$, we have,

$$\nabla_x \cdot \langle A(y)\left(I_n + \nabla_y v^T(y)\right)\rangle \nabla_x u^0 = f.$$

Now define the effective conductivity to be

$$A^0 := \langle A(y)\left(I_n + \nabla_y v^T(y)\right)\rangle$$

Now, A^0, u^0 and u^1 are determined by the following system of equations:

$$\begin{cases} \begin{cases} \nabla_y \cdot A(y)\left(I_n + \nabla_y v(y)^T\right) = 0, \\ v \in (H^1(Q_p))^n. \end{cases} \\ \\ A^0 := \langle A(y)\left(I_n + \nabla_y v^T(y)\right)\rangle \\ \\ \begin{cases} \nabla_x \cdot A^0 \nabla_x u^0(x) = f(x),\ u^0 \in H^1(\Omega). \\ u^0(x)|_{\partial\Omega} = g(x) \end{cases} \\ \\ u^1(x,y) = v(y)^T \nabla_x u^0(x) \end{cases} \tag{0.7}$$

The first equation is called the **"cell-problem"**. The approximation to u^ϵ, denoted ϕ^ϵ, is given by

$$\phi^\epsilon(z) = u^0(z) + \epsilon u^1(z, \frac{z}{\epsilon}).$$

A helpful rule for calculating A^0

It is helpful to have a rule for calculating the effective conductivity, A_0 (this is found in [1])

LEMMA **0.1.** *If u is any Q-periodic potential function ($u = \lambda + \nabla v$ for some v is Q-periodic) and $\nabla \cdot (A(z)u(z)) = 0$, then*

$$A^0 \langle u \rangle = \langle Au \rangle$$

where

$$\langle u \rangle = \frac{1}{|Q|} \int_Q u(z) dz.$$

I give this without proof. However, it is not random. The function given by the cell problem,

$$u = I_n + \nabla_y v(y)^T,$$

is a periodic potential function with $\nabla_y \cdot A(y)u = 0$ and $\langle u \rangle = I_n$. We also see by the system of equations (0.7), that

$$A^0 \langle u \rangle = A^0 = \langle A(y)u \rangle$$

0.1 An example in 1D

Let $a(z)$ be a real scalar valued $[0,1]$-periodic function which represents the conductivity tensor in the following problem:

$$\begin{cases} \frac{d}{dz}\left(a(\frac{z}{\epsilon}) \frac{du^\epsilon(z)}{dz} \right) = f, \ z \in [0,1], \\ u^\epsilon(0) = a, \\ u^\epsilon(1) = b. \end{cases}$$

For some $f \in L^2[0,1]$ and $a, b \in \mathbb{R}$. Assume $\exists C > c > 0$ such that $\forall z \in [0,1]$, $c \leq a(z) \leq C$. We will use lemma 0.1 to calculate the effective tensor, a^0. Consider the function $u(y) = \frac{1}{a(y)}$.

$$u(y) = \frac{d}{dy} \int_0^y \frac{1}{a(y)} dy,$$

(i.e. u is a potential function), and

$$\frac{d}{dy}\left(a(y)u(y) \right) = \frac{d}{dy} 1 = 0.$$

Thus, u satisfies the assumptions in the lemma. Then, by the lemma and the definition of u,

$$a^0 \langle u(y) \rangle = \langle a(y)u(y) \rangle = \langle \frac{a(y)}{a(y)} \rangle = \langle 1 \rangle = 1.$$

Thus,

$$a^0 = \frac{1}{\langle u(y) \rangle} = \frac{1}{\langle \frac{1}{a(y)} \rangle} = \langle a(y)^{-1} \rangle^{-1}.$$

The effect conductivity tensor is then the harmonic mean of the original conductivity tensor.

Problem set

(1) **Find the effective conductivity tensor** Consider a periodic conductivity of the form

$$A(z) = \begin{pmatrix} a_1(z_1) & 0 \\ 0 & a_2(z_1) \end{pmatrix} \quad \text{where} \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

This represents materials such as those with cell structure (B); although here $a_i(z)$ is smooth. The goal of this problem is to use Lemma 0.1 to calculate A_0.
Assume that A^0 is a constant diagonal 2×2 matrix and assume $A(z)$ is smooth.

(a) Show for all $u(z) = u(z_1)$, $\nabla \cdot Au(z) = \partial_1 (a_1(x_1)u_1(x_1))$.

(b) Find a $u(z)$ such that $u_1(z_1)a(z_1)$ is constant and $u(z)$ is a potential. Follow the 1D example for this.

(c) Calculate A^0.

(2) **Calculate the approximation** The goal of this problem is to compare the explicit solution of

$$\begin{cases} \frac{d}{dz}\left((a\left(\frac{z}{\epsilon}\right)\frac{d}{dz}u^\epsilon(z)\right) = 0, \ z \in [0,1] \\ u^\epsilon(0) = 0, \\ u^\epsilon(1) = 1, \end{cases}$$

to $\phi^\epsilon(z) = u^0(z) + \epsilon u^1(z, \frac{z}{\epsilon})$, where u^0, u^1 are given by the following system:

$$\begin{cases} \begin{cases} \frac{d}{dy}\left(a(y)\left(1 + \frac{d}{dy}v(y)\right)\right) = 0, \ y \in [0,1] \\ v \text{ is } [0,1]\text{-periodic.} \end{cases} \\ \\ a^0 := \langle a(y)\left(1 + \frac{d}{dy}v(y)\right)\rangle \\ \\ \begin{cases} a^0 \frac{d^2}{dx^2}u^0(x) = 0, \ x \in [0,1]. \\ u^0(0) = 0, \\ u^0(1) = 1. \end{cases} \\ \\ u^1(x,y) = v(y)\frac{d}{dx}u^0(x) \end{cases}$$

$\langle \cdot \rangle$ is the average over $[0,1]$. Derive the explicit formulas for u^ϵ and ϕ_ϵ and show the following convergence holds:

$$\|u^\epsilon - u^0\|_{L^2(0,1)} \to_{\epsilon \to 0^+} 0$$
$$\left\| \frac{d}{dz}u^\epsilon - \frac{d}{dz}\phi^\epsilon \right\|_{L^2(0,1)} \to_{\epsilon \to 0^+} 0.$$

It is assumed that $a(z)$ is $[0,1]$-periodic and $C^1(\mathbb{R})$ and $\exists C > c > 0$ such that $c < a(z) < C$. $a^0 = \langle a(y)^{-1}\rangle^{-1}$ is given, but it doesn't matter as it can be factored out of the third equation in the system.

You will need the following fact about the convergence of averages:

LEMMA **0.2.** *Let $a(y)$ be a bounded measurable $[0,1]$-periodic function on $\mathbb{R}$, then*

$$\left\| \int_0^z a\left(\frac{t}{\epsilon}\right) dt - z \int_0^1 a(y)dy \right\|_{L^2(0,1)} \to_{\epsilon \to 0^+} 0.$$

You do NOT need to prove this; however, it is not complicated to prove and in doing so you can get a convergence rate.

[1] V.V. Jikov, S.M. Kozlov, O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals. Springer-Verlag, Berlin, 1994.

[2] E, Weinan, Principles of Multiscale Modeling, Cambridge University Press, Cambridge, 2011.

[3] Y. Grabovsky, Methods in Applied Mathematics (Lecture Notes).