Problem Set 5 by Nicole Zalewski

(Out Wed 10/23/2024, Due Wed 10/30/2024)

Problem 1

Definition 1. *For a given $\phi(v)$ and solution to the Boltzmann equation $f = f(t,x,v)$, ϕ is said to induce a* conserved quantity $\int_X \int_V \phi(v) f(t,x,v) dv dx$ *if*

$$\frac{d}{dt} \int_X \int_V \phi(v) f(t,x,v) dv dx = 0$$

Definition 2. *We call $\phi(v)$ a* collision invariant *if*

$$\int_{\mathbb{R}^3} \phi(v) Q(f,f)(t,x,v) dv = 0$$

for all densities f, with Q(f,f) being the collision operator given in class

(1) Suppose $\lim_{|x|\to\infty} f(t,x,v) \equiv 0$. Show that $\phi(v)$ being a collision invariant means there is a conserved quantity.

(2) Prove the following equivalent definition of a collision invariant:

Lemma 1. *The function ϕ is a collision invariant, if and only if for all v, $w \in \mathbb{R}^3$ and all $\Omega \in S^2$ we have*

$$\phi(v) + \phi(w) = \phi(v') + \phi(w')$$

(3) Prove that a distribution f is an equilibrium distribution if it is an exponential of a collision invariant. What does this tell you about Maxwellians, as defined in class?

(4) Explain why if $\phi(v) \in \text{span}\{1, v_1, v_2, v_3, |v|^2\}$, then ϕ is collision invariant. (In fact, the converse is also true). What are the physical interpretations of the conserved quantities associated to the spanning vectors $1, v_i$, and $|v|^2$? Derive the local conservation laws for these quantities using the Boltzmann equation and proven facts.

Instructions

Email your solutions (i.e., a scan or typed version of your pen-and-paper part; and programming codes in a way that they can be run by someone else) to `nicole.zalewski@temple.edu` with the email subject `Math 8200. Homework 5` and all the submitted filenames starting with your family name.