Temple Math 8200 **Multiscale Modeling and Methods** **Fall 2024**

Problem Set 1 by Henry Brown

(Out Mon 09/23/2024, Due Mon 09/30/2024)

Problem 1

Find the effective conductivity tensor Consider a periodic conductivity of the form

$$A(z) = \begin{pmatrix} a_1(z_1) & 0 \\ 0 & a_2(z_1) \end{pmatrix} \quad \text{where} \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

This represents laminate materials such as those with cell structure (B) from page 2 of the lecture notes; although here $a_i(z)$ is assumed to be smooth. The goal of this problem is to use Lemma 0.1 from the notes to calculate A_0.
<u>Assume that A^0 is a constant diagonal 2×2 matrix</u>.

1. Show for all $u(z) = u(z_1)$, $\nabla \cdot Au(z) = \partial_1 \left(a_1(x_1)u_1(x_1)\right)$.
2. Find a $u(z)$ such that $u_1(z_1)a(z_1)$ is constant and $u(z)$ is a potential. Follow the 1D example in the lecture notes for this.
3. Calculate A^0.

Problem 2

Calculate the approximation The goal of this problem is to compare the explicit solution of

$$\begin{cases} \frac{d}{dz}\left(\left(a\left(\frac{z}{\epsilon}\right)\frac{d}{dz}u^\epsilon(z)\right) = 0,\ z \in [0,1] \\ u^\epsilon(0) = 0, \\ u^\epsilon(1) = 1, \end{cases}$$

to $\phi^\epsilon(z) = u^0(z) + \epsilon u^1(z, \frac{z}{\epsilon})$, where u^0, u^1 are given by the following system:

$$\begin{cases} \begin{cases} \frac{d}{dy}\left(a(y)\left(1 + \frac{d}{dy}v(y)\right)\right) = 0,\ y \in [0,1] \\ v \text{ is } [0,1]\text{-periodic.} \end{cases} \\ \\ a^0 := \langle a(y)\left(1 + \frac{d}{dy}v(y)\right)\rangle \\ \\ \begin{cases} a^0 \frac{d^2}{dx^2}u^0(x) = 0,\ x \in [0,1]. \\ u^0(0) = 0, \\ u^0(1) = 1. \end{cases} \\ \\ u^1(x,y) = v(y)\frac{d}{dx}u^0(x) \end{cases}$$

$\langle \cdot \rangle$ is the average over $[0,1]$. Derive the explicit formulas for u^ϵ and ϕ_ϵ and show the following convergence holds:

$$\|u^\epsilon - u^0\|_{L^2(0,1)} \to_{\epsilon\to 0^+} 0$$
$$\left\|\frac{d}{dz}u^\epsilon - \frac{d}{dz}\phi^\epsilon\right\|_{L^2(0,1)} \to_{\epsilon\to 0^+} 0.$$

It is assumed that $a(z)$ is $[0,1]$-periodic and $C^1(\mathbb{R})$ and $\exists C > c > 0$ such that $c < a(z) < C$. $a^0 = \langle a(y)^{-1}\rangle^{-1}$ is given, but it doesn't matter as it can be factored out of the third equation in the system.
You will need the following fact about the convergence of averages:

LEMMA **.1.** *Let $a(y)$ be a bounded measurable $[0,1]$-periodic function on $\mathbb{R}$, then*

$$\left\|\int_0^z a\left(\frac{t}{\epsilon}\right)dt - z\int_0^1 a(y)dy\right\|_{L^2(0,1)} \to_{\epsilon\to 0^+} 0.$$

You do NOT need to prove this; however, it is not complicated to prove, and, in doing so, you can get a convergence rate.

Instructions

Email your solutions (i.e., a scan or typed version of your pen-and-paper part; and programming codes in a way that they can be run by someone else) to `henry.brown0001@temple.edu` with the email subject `Math 8200. Homework 1` and all the submitted filenames starting with your family name.