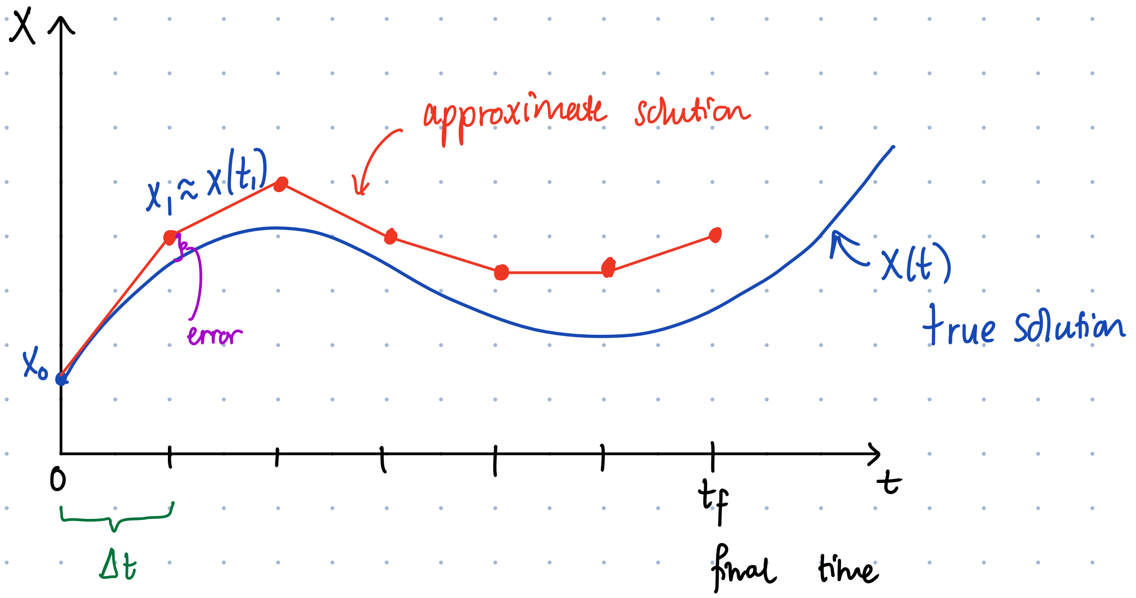
ODE Approximation

$$\left.\begin{aligned} x'(t) &= f(x) \\ x(t_0) &= x_0 \quad \text{initial condition} \end{aligned}\right\} \text{initial value problem}$$

Consider the autonomous case. Here $t_0 = 0$.

Want to numerically approximate solution of ODE



Consider n steps then $n\,\Delta t = t_f$

$$t_j = j\,\Delta t \qquad j = 1, \ldots, n$$

Solution to differential equation is an unknown curve/function that depends on time

Differential equation gives formula for computing slope of tangent line at any point on the unknown curve

We are given the initial condition - one point on the solution curve & the task is to find successive points on the curve.

Consider the slope of the tangent line at

x_0. It is given by $f(x_0)$.

Euler Method for approximating ODEs says we can

find an approximate value of x at $t_1 = \Delta t$

i.e $x_1 \approx x(t_1)$ by assuming $\frac{x_1 - x_0}{\Delta t} = f(x_0)$

So $x_1 = x_0 + \Delta t \cdot f(x_0)$

To find x_2 we assume x_1 really is equal to $x(t_1)$ then

$$x_2 = x_1 + \Delta t \cdot f(x_1)$$

Euler Method does this approximation for the rest of the points

General formula : Forward Euler

$$x_j = x_{j-1} + \Delta t \cdot f(x_{j-1}) \qquad j=1,\dots,n$$

$$x_j \approx x(t_j) = x(j\Delta t)$$

What is the error made in one step of Euler method?

Suppose $x(t_{j-1}) = x_{j-1}$

approx $\quad x_j = x_{j-1} + \Delta t \cdot f(x_{j-1})$

true $\quad x(t_j) = x(t_{j-1} + \Delta t)$

$$= x(t_{j-1}) + \Delta t * x'(t_{j-1}) + \frac{1}{2}(\Delta t)^2 x''(t_{j-1}) + O(\Delta t^3)$$

$$e_j = x(t_j) - x_j = \frac{1}{2}(\Delta t)^2 x''(t_{j-1}) + \underbrace{O(\Delta t^3)}$$

error at step j

higher order terms become insignificant as Δt becomes smaller

There are $n = \frac{t_f}{\Delta t}$ steps $\Rightarrow O\left(\frac{1}{\Delta t}\right)$ steps

Per step accumulate error $O(\Delta t^2)$

Global error → $O\left(\frac{1}{\Delta t}\right) O(\Delta t^2) = O(\Delta t)$

Euler Method is a first order method

Reduce time step size Δt by 2, error goes down a factor of 2.

We can do better for example

1. RK2 - Runge-Kutta 2 is a second order scheme $O(\Delta t^2)$

$E \sim O(\Delta t^2) \Rightarrow$ halve $\Delta t \Rightarrow$ error goes down by factor of 4

$E_{\Delta t} = O(\Delta t^2)$

$E_{\frac{\Delta t}{2}} = O\left(\left(\frac{\Delta t}{2}\right)^2\right) = O\left(\frac{\Delta t^2}{4}\right)$

2. RK4 is a fourth order scheme $O(\Delta t^4)$
   halve $\Delta t \Rightarrow$ error goes down by factor of 16!

Better accuracy

COVID-19 Modeling

Model how disease spreads

1. Agent Based Approach

think contagion code

Represent humans as agents and simulate what happens when they interact and possibly pass disease on

The ABM can include characteristics of disease

1. probability of infection
2. likelihood of them staying infected
3. amount of time needed to recover

ABM can also build in agent attributes to match population demographics like

age, profession, student/nonstudent, where someone lives etc.

Build a simulation then try add in policy interventions to see effects

i) Mask wearing
ii) Social distancing
iii) Quarantine / Isolation
iv) Vaccination (hard)

2. Differential Equation Models

Don't resolve the population to its finer details instead group into broad classes eg.

SIR model Simple case

$S(t)$	number of susceptible individuals
$I(t)$	number of infected individuals
$R(t)$	number of recovered individuals

Suppose we have a total population N we can also define fractions of total population

$$s(t) = \frac{S(t)}{N}$$

$$i(t) = \frac{I(t)}{N}$$

$$r(t) = \frac{R(t)}{N}$$

The differential modeling idea is to figure out how $s(t)$, $i(t)$, $r(t)$ vary in time.

Note that $s(t) + i(t) + r(t) = 1$

$$\frac{ds}{dt} = -b\,s(t)\,i(t)$$

$$\frac{dr}{dt} = k\,i(t)$$

$$\frac{di}{dt} = b\,s(t)\,i(t) - k\,i(t)$$

b – constant, probability of infection
k – constant, probability of recovery

Assumptions

1. no one added to susceptible group (no birth, no immigration)
2. leave susceptible group by becoming infected after interaction with infected
3. assume a certain fraction of infected population recovers

Missing -> death rate

Implement model for $t_f = 140$ days

with $s(0) = 1$, $i(0) = 1.27 \times 10^{-6}$, $r(0) = 0$
$b = 1/2$, $k = 1/3$, $\Delta t = 10$