Population Dynamics Theory

Study of why & how populations change in size & nature over time.

Simple model

Deterministic, non-agent based

Consider a single specie denoted by $x(t)$

$x(t)$ — population at time t

Model rate of change of population

$$\dot{x}(t) = \beta x(t) - \delta x(t)$$

$x'(t)$

$\frac{dx}{dt}$

birth rate   death rate

rate of change of population is due to some of the population getting born and some dying

$$\dot{x}(t) = (\beta - \delta) x(t)$$

$$= \lambda x(t)$$

↳ growth rate

$\lambda \in \mathbb{R}$

$$\dot{x} = \lambda x \qquad \text{ODE}$$
$$x(0) = x_0 \qquad \text{initial condition}$$

Has unique solution

$$\frac{dx}{dt} = \lambda x$$

$$\int \frac{1}{x}\,dx = \int \lambda\, dt$$

$$\ln x = \lambda t + C$$

$$e^{\ln x} = e^{(\lambda t + C)}$$

$$x = e^{\lambda t} \underbrace{e^{C}}_{K}$$

Know $x(0) = x_0 = K$

$$x(t) = x_0 e^{\lambda t}$$

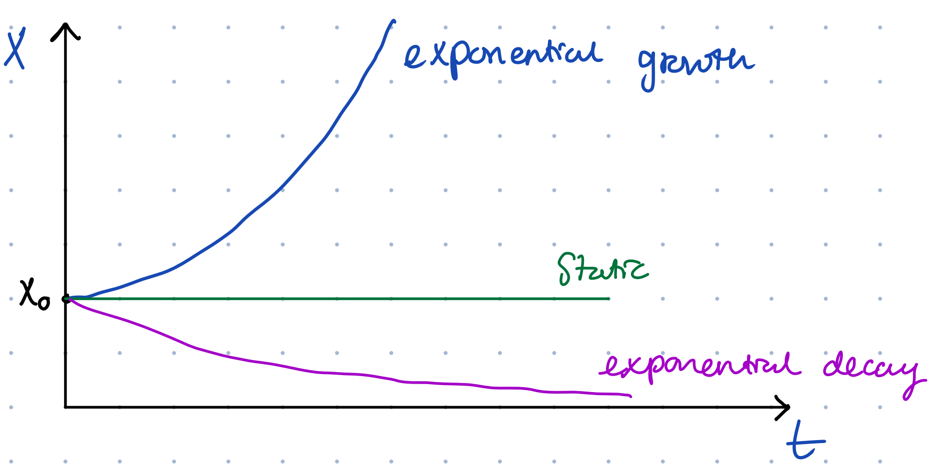
See for different values of lambda

Dynamics for

$\lambda > 0$

$\lambda = 0$

$\lambda < 0$

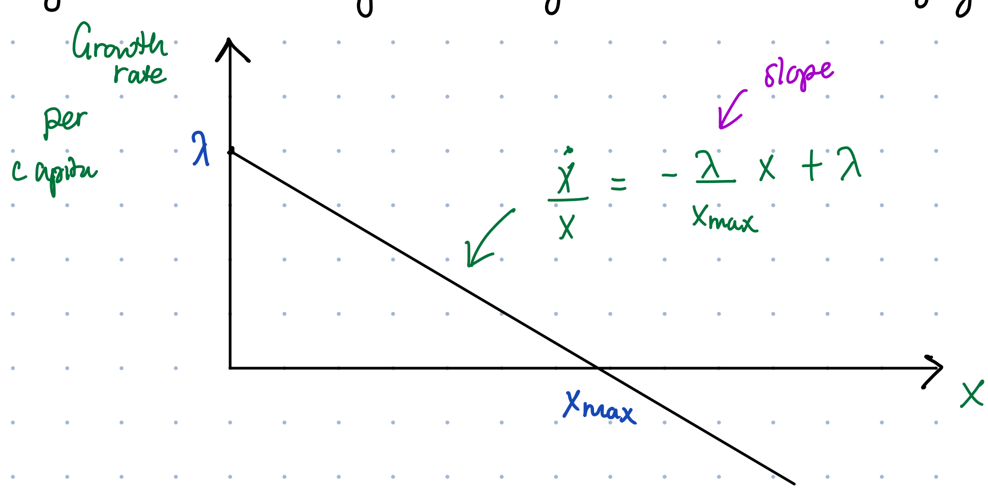


In reality for $\lambda > 0$ you don't get exponential growth. There is limited resources - space
- food etc.

To model effects of overcrowding & limited resources biologist & demographers assume when x get sufficiently large the per capita growth rate $\frac{\dot{x}}{x}$ decreases.

$\frac{\dot{x}}{x} = \lambda$

Say x can grow only to x_{max} (carrying capacity)



$$\frac{\dot{x}}{x} = -\frac{\lambda}{x_{max}}x + \lambda$$

for populations greater than x_{max}, growth rate negative

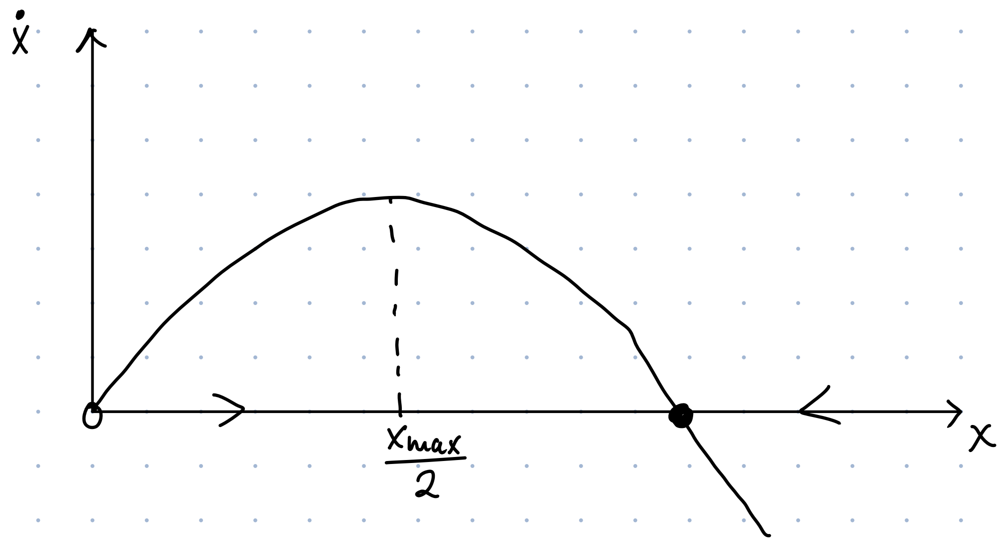
$\frac{\dot{x}}{x}$ decreases linearly with x

$$\frac{\dot{x}}{x} = -\frac{\lambda}{x_{max}} x + \lambda$$

$$\dot{x} = -\underbrace{\frac{\lambda}{x_{max}}}_{\substack{\text{const} \\ \mu}} x^2 + \lambda x = \lambda x \left(1 - \frac{1}{x_{max}}\right)$$

$$\dot{x}(t) = \lambda x - \mu x^2$$ logistic equation

The equation can be solved analytically but we can look at the vector field to see dynamics too. Plot $\dot{x}$ vs. x



fixed points $\dot{x} = 0$

$x = 0$ or $x = x_{max}$

$x = 0$ unstable fixed point

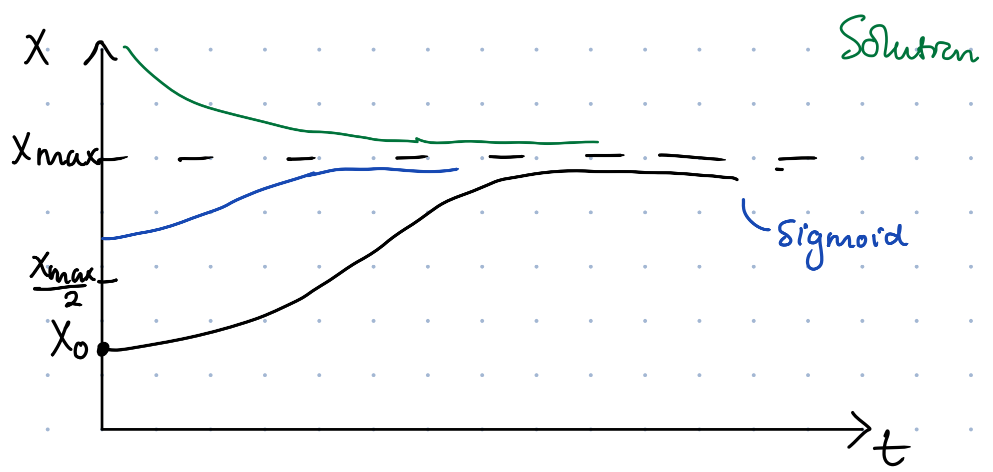
a small population will grow exponentially fast & run away from $x=0$

$x = x_{max}$ stable fixed point

If x is disturbed slightly from x_{max}

$x(t) \to x_{max}$

The population always approaches the carrying capacity



Unless there is no one around to start reproducing, then for $x_0 = 0$, $x(t) = 0$ for all time

Two species

1) Lotka-Volterra predator-prey

$$\dot{x} = \underbrace{\alpha \cdot x}_{\text{birth}} - \underbrace{\beta x y}_{\text{death by predators}} = x(\alpha - \beta y)$$

$$\dot{y} = \underbrace{\gamma x y}_{\text{birth by eating}} - \underbrace{\delta y}_{\text{death}} = y(\gamma x - \delta)$$

ODE system of equations
Nonlinear

$\alpha, \beta, \gamma, \delta > 0$

Unrealistic assumptions?

1) prey has enough food at all times
2) food of predator depends on prey only
3) rate of change of population proportional to size
4) predators have limitless appetite