Temple Math 5044 **Introduction to Numerical Analysis II** **Spring 2023**

Problem Set 1

(Out Thu 01/19/2023, Due Thu 02/02/2023)

Problem 1

Consider a Lotka-Volterra predator-prey model for a population of carps and pikes, whose numbers are given by $u(t)$ and $v(t)$, respectively. The growth/decay rates are given by

$$\begin{aligned} \frac{du}{dt} &= \quad u - 4uv \;, \\ \frac{dv}{dt} &= -v + 5uv \;. \end{aligned} \tag{1}$$

(a) Show that the function $H(u,v) = uv\ \exp(-5u-4v)$ is a *constant of motion*, i.e. if $(u(t),v(t))$ is a solution of (2), then $H(u(t),v(t))$ is constant in time.

(b) Using Matlab's `mesh` (or similar) command, plot the function H on the domain $(u,v) \in [0,1]^2$.

(c) Using Matlab's `quiver` command, plot the velocity field given by the right hand side vector of (2), scaled to length 1 everywhere. Overlay the quiver plot by suitable isocontours of the function H, using Matlab's `contour` command.

(d) Starting with $u(0) = 0.2$ and $v(0) = 0.8$, approximate (2) using Euler's method for $t \in [0,8]$. Use steps of size $\Delta t = 0.01$. Plot all 801 points obtained from this numerical solution into the figure created in (c). Explain why the resulting curve is not closed.

(e) Run the same computation with $\Delta t = 0.02$ and $\Delta t = 0.04$, as well as $\Delta t = 0.005$ and $\Delta t = 0.0025$. How does the observed approximation error behave. Explain your observations.

Problem 2

Consider the Lotka-Volterra predator-prey model from Problem 1:

$$\frac{d}{dt}\vec{u}(t) = \vec{f}(\vec{u}(t)) \;, \tag{2}$$

where $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\vec{f}(\vec{u}) = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix} = \begin{pmatrix} u - 4uv \\ -v + 5uv \end{pmatrix}$.

(a) Calculate the Jacobian matrix $D\vec{f}(\vec{u}) = \begin{pmatrix} \frac{\partial f_1}{\partial u}(u,v) & \frac{\partial f_1}{\partial v}(u,v) \\ \frac{\partial f_2}{\partial u}(u,v) & \frac{\partial f_2}{\partial v}(u,v) \end{pmatrix}$, and the product $D\vec{f}(\vec{u}) \cdot \vec{f}(\vec{u})$.

(b) Use the expressions from (a) to formulate a second order Taylor series method for (2).

(c) Start with the results produced in parts (d) and (e) in Problem 1, and add the analogous plots produced with the Taylor series method formulated above (using the same parameters, but using different colors, line styles, and/or labels for the two methods). Explain your observations.

(d) Use the Taylor series method to approximate the time T that it takes for a trajectory $\vec{u}(t)$ to return (for the first time) to its initial value, i.e. $\vec{u}(T) = \vec{u}(0)$. Do this for at least 50 initial values

$$\vec{u}(0) = \begin{pmatrix} a \\ 0.25 \end{pmatrix} \text{ with } 0 < a < 0.2$$

and thus produce a plot of the function $T(a)$.

(e) Explain the fact that for $a \approx 0.25$, it takes close to $T = 2\pi$ for the solution to "go around" once.

Instructions

For each problem set, you need to submit one document, either in class or via email to the course instructor, that contains plots and explanations (hand-written or typed). If you decide to email the document, name it `yourfamilyname_problemset1.pdf`, where 1 stands for the number of the problem set.

In addition, for each programming task, email your respective program to the course instructor, under the filename `yourfamilyname_problem1a.m`, where `1` stands for the problem number and `a` for the sub-problem letter. [On this problem set, you need to submit four Matlab codes, for `1b`, `1c`, `1d`, `1e`, and `2c`, and `2d`.]