Temple 8024 | **Numerical Differential Equations II** | **Spring 2017**

Problem Set 1

(Out Thu 01/26/2017, Due Thu 02/09/2017)

Problem 1

Consider the linear advection equation

$$\begin{cases} u_t + u_x = 0 \\ u(x,t) = u_0(x) \end{cases}$$

on $x \in [0,1]$ with periodic boundary conditions, and $t \in [0,1]$, with discontinuous initial data

$$u_0(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}[\\ 1 & x \in [\frac{1}{2}, 1[\end{cases}$$

Implement the three schemes (UW) upwind, (LF) Lax-Friedrichs, and (LW) Lax-Wendroff, and perform a numerical error analysis with $k = 0.9\,h$. Specifically ...

(1) find the convergence rates of the three schemes when measuring the error in $\|\cdot\|_\infty$ and $\|\cdot\|_1$.

(2) compare these convergence rates to the ones observed when changing the initial conditions to

  (a) $u_0(x) = \sin(2\pi x)$

  (b) $u_0(x) = \begin{cases} x & x \in [0, \frac{1}{2}[\\ 1 - x & x \in [\frac{1}{2}, 1[\end{cases}$

Explain your observations.

Problem 2

Solve the following three equations:

(1) Inviscid Burgers' equation

$$\begin{cases} u_t + uu_x = 0 \text{ on } (x,t) \in]0,1[\times]0,1[\\ u(0,t) = 1 \\ u(x,0) = \cos^2(\frac{3\pi}{2}x)\exp(-2x) \end{cases}$$

with a non-conservative upwind method

$$\frac{U_j^{n+1} - U_j^n}{k} + U_j^n \frac{U_j^n - U_{j-1}^n}{h} = 0\ .$$

(2) Inviscid Burgers' equation

$$\begin{cases} u_t + \left(\frac{1}{2}u^2\right)_x = 0 \text{ on } (x,t) \in]0,1[\times]0,1[\\ u(0,t) = 1 \\ u(x,0) = \cos^2(\frac{3\pi}{2}x)\exp(-2x) \end{cases}$$

with a conservative upwind method

$$\frac{U_j^{n+1} - U_j^n}{k} + \frac{\frac{1}{2}(U_j^n)^2 - \frac{1}{2}(U_{j-1}^n)^2}{h} = 0\ .$$

(3) Viscous Burgers' equation

$$\begin{cases} u_t + uu_x = 10^{-3} u_{xx} \text{ on } (x,t) \in]0,1[\times]0,1[\\ u(0,t) = 1 \ , \ u(1,t) = 0 \\ u(x,0) = \cos^2(\frac{3\pi}{2}x)\exp(-2x) \end{cases}$$

with a non-conservative treatment of the nonlinear advection

$$\frac{U_j^{n+1} - U_j^n}{k} + U_j^n \frac{U_j^n - U_{j-1}^n}{h} = 10^{-3} \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2} \ .$$

(I strongly recommend an implicit treatment of the diffusion term.)

For the resolutions $h \in \{\frac{1}{100}, \frac{1}{500}, \frac{1}{2500}\}$, plot the solutions at $t \in \{0.1, 0.5, 1.0\}$. Carefully explain your observations.