Temple 3044 **Numerical Analysis II** **Spring 2012**

Problem Set 7

(Out Thu 03/20/2012, Due Thu 03/29/2012)

Instructions

- Problems marked with **(T)** are theory problems. Their solutions are to be submitted on paper.
- Problems marked with **(P)** are practical problems, and require the use of the computer. Their solutions are to be submitted on paper, and usually require two parts: (a) a description of the underlying theory; and (b) code segments, printouts of program outputs, plots, and whatever it required to convince the grader that you have understood the theory and addressed all practical challenges appropriately.

Generally, naked numbers are not acceptable. Solutions must include a short write-up describing the problem, your solution technique, and procedural details. To include a computer printout use the cut and paste method for placement of materials in your work. All things must be clearly labeled.

Problem H

(T)&(P) The evolution of a solvent concentration u in a channel flow of velocity 1 can be described by the inhomogeneous convection-diffusion equation

$$u_t + u_x = \varepsilon u_{xx} + 1 \ , \tag{1}$$

where ε is the solvent's diffusivity, and the source 1 denotes the insertion (from the top) of new solvent into the channel, at a rate that is uniform in space and constant in time. Consider the channel be at $x \in [0, 1]$, and zero inflow, i.e. $u(0) = 0$. Solutions of (1) converge (in time) to the steady state ODE

$$-\varepsilon u_{xx} + u_x = 1 \ . \tag{2}$$

Consider the following two cases:

(A) small diffusion: $\varepsilon \ll 1$, where $u(1) = 0$, and (B) no diffusion: $\varepsilon = 0$.

(a) For both cases (A) and (B), calculate the true solution $u(x)$ of (2).

(b) Plot the solution of (A) for $\varepsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$, and the solution of (B). Describe what happens as $\varepsilon \to 0$. Provide an explanation for the observed phenomenon.

(c) Write a Matlab program that approximates the solution of (A) using finite differences on a regular grid. For $\varepsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$, find the maximum mesh resolution $h = h(\varepsilon)$ that is required to approximate the true solution within 1% accuracy (relative error, measured in the maximum norm).

(d) Write a new Matlab program that approximates the solution of (A) using finite differences on a so-called *Shishkin mesh*, i.e., a grid that is piecewise uniform, with two different resolutions. Specifically, based on the plots obtained in (b), find a suitable point $z = z(\varepsilon) \in (0, 1)$, such that a finite difference approximation[1] of (2) on the Shishkin mesh $X = \{0, h_1, 2h_1 \dots, z - h_1, z, z + h_2, \dots, 1 - h_2, 1\}$ with $h_2 \ll h_1$ yields a numerical approximation within 1% accuracy with significantly fewer points than used in (c). In fact, for each ε, try to achieve the target accuracy with as few grid points as possible.

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[1] At $x = z$, you must derive a non-equidistant approximation to u_{xx}, using the three points $z - h_1$, z, and $z + h_2$.

Problem I

(P) Modifying the program `temple3044_poisson.m` on the course website, solve the following 2D Poisson problem

$$\begin{cases} -\Delta u(x,y) = f(x,y) & \text{in } [0,1]\times[0,1] \\ u = 0 & \text{on } \{0,1\}\times[0,1]\cup[0,1]\times\{0,1\} \end{cases}$$

where

$$f(x,y) = \begin{cases} 10 & \text{if } (x,y)\in[0,\tfrac{1}{2}]\times[0,\tfrac{1}{2}] \\ 0 & \text{otherwise} \end{cases}$$

numerically, and report the values

$$u(\tfrac{1}{2},\tfrac{1}{2})$$

and

$$\max_{0\le x,y\le 1} u(x,y)$$

with at least three digits of accuracy (ensure, by testing various grid resolutions, that your results possess this desired accuracy).