Temple 3044 **Numerical Analysis II** **Spring 2012**

Problem Set 6

(Out Thu 02/23/2012, Due Thu 03/01/2012)

Instructions

- Problems marked with **(T)** are theory problems. Their solutions are to be submitted on paper.
- Problems marked with **(P)** are practical problems, and require the use of the computer. Their solutions are to be submitted on paper, and usually require two parts: (a) a description of the underlying theory; and (b) code segments, printouts of program outputs, plots, and whatever it required to convince the grader that you have understood the theory and addressed all practical challenges appropriately.

Generally, naked numbers are not acceptable. Solutions must include a short write-up describing the problem, your solution technique, and procedural details. To include a computer printout use the cut and paste method for placement of materials in your work. All things must be clearly labeled.

Problem F

(P)&(T) Consider the linear ODE system

$$\begin{cases} \vec{u}'(t) = A \cdot \vec{u}(t) \\ \vec{u}(0) = \mathring{\vec{u}} \end{cases} \tag{1}$$

where

$$A = \begin{pmatrix} -50000 & 49999 & 0 & 0 \\ 49999 & -50000 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 10 & 0 \end{pmatrix} \quad \text{and} \quad \mathring{\vec{u}} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

We would like to approximate the solution of (1) at $\vec{u}(1)$, using an ODE solver with equidistant time steps. We are happy to be within 5% accuracy.

(1) Calculate the true solution $\vec{u}(1)$.

(2) Find numerically the maximum time step that yields the desired accuracy, using ...

  (a) forward Euler

  (b) backward Euler

  (c) RK4

  (d) Crank-Nicolson

(3) Explain your observations.

(4) Plot the approximate solution obtained by backward Euler (with maximum admissible time step), together with the true solution.

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Problem G

(P)&(T) Consider the 1d Poisson equation

$$\begin{cases} -u_{xx} = f & \text{in }]0,1[\\ u = 0 & \text{on } \{0,1\} \end{cases} \tag{2}$$

with $f(x) = \sin(\phi(x))(\phi_x(x))^2 - \cos(\phi(x))\phi_{xx}(x)$, where $\phi(x) = 9\pi x^2$.
Run the program `mit18336_poisson1d_error.m` given on the course web site, which approximates (2) by a sequence of linear systems, based on the approximation $u_{xx} \approx D^2 u$,
where $D^2 u(x) = \frac{1}{h^2}(u(x+h) - 2u(x) + u(x-h))$.

(1) Explain the observed error convergence rate.

(2) Return to the original system matrix based on $u_{xx} \approx D^2 u$. Now change the right hand side vector from $f_i = f(ih)$ to $f_i = f(ih) + \frac{h^2}{12} D^2 f(ih)$. Prove that this modification yields fourth order accuracy, and produce an error convergence plot that verifies this result.[1] How does the error constant compare to fourth order system matrix in part (2)?

(3) Change the right hand side to

$$f(x) = \begin{cases} 1 & \text{for } x \le \frac{1}{2} \\ 0 & \text{for } x > \frac{1}{2} \end{cases}$$

and report and explain the new error convergence rate for the two solution approaches.

[1]This trick is called *deferred correction*.