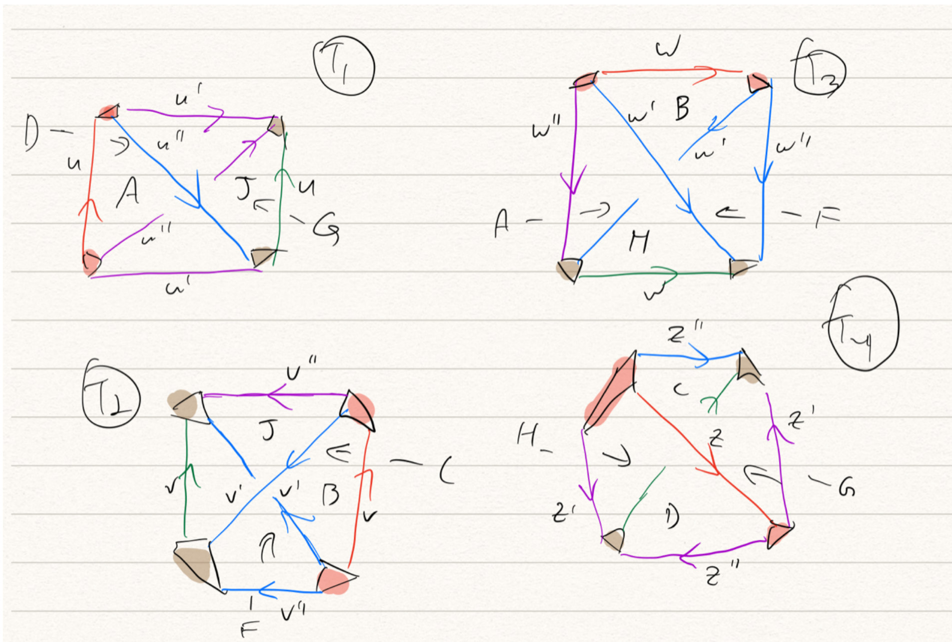
Math 9510 In-Class Worksheet

For Tuesday & Thursday, Feb 7–9, 2023

Note: Page and exercise numbers from Purcell refer to the arXiv version of the book, available at `https://arxiv.org/abs/2002.12652`

1. For the 4–tetrahedron triangulation of the Whitehead link pictured below, verify that if the gluing equations around the red and blue edges are satisfied, then *all* of the gluing equations are satisfied.



2. For the 4–tetrahedron triangulation of the Whitehead link, work out the completeness equations in the two cusps. Note that the invariant $H(\alpha)$ associated to a curve α in the cusp torus is carefully defined in Definition 4.12 and equation (4.2) of Purcell's book.

3. If α and β are two curves in a cusp torus T that generate $\pi_1(T)$, show that the holonomy of α is parabolic if and only if the holonomy of β is parabolic. Conclude that $H(\alpha) = 1$ if and only if $H(\beta) = 1$. (Does this work for the Whitehead link?)

Next, we will generalize this picture and guess the dimension of the space of solutions for a general triangulation.

4. Let $\overline{M}$ be a compact 3–manifold with boundary consisting of tori. Let $\mathcal{T}$ be an ideal triangulation of M, the interior of $\overline{M}$. Prove that the number of tetrahedra in $\mathcal{T}$ is equal to the number of edges.

Hint: consider the two-dimensional combinatorics of $\partial\overline{M}$. Each boundary torus is tiled by triangles that truncate the ideal vertices of tetrahedra (for instance, the little pink and brown triangles in the above picture for the Whitehead link). Relate the number of vertices, edges, and triangles in this boundary tiling to the counts of tetrahedra in $\mathcal{T}$.

Based on Exercise 4, we know that the edge gluing equations for $\mathcal{T}$ have n variables (one per tetrahedron) and n equations (one per edge). If all of the equations were independent, we would expect the solution set to be zero-dimensional. However, there are always redundancies:

5. Let $\overline{M}$ be a compact 3–manifold with boundary consisting of $k \geq 1$ tori. Let $\mathcal{T}$ be an ideal triangulation of M, the interior of $\overline{M}$.

a) Prove that at least one edge equation is always redundant.

b) Prove that at least k edge equations are always redundant. (This is what happened with the Whitehead link.)

Based on Exercise 5, we expect the solution set to the edge gluing equations to have complex dimension k (one complex degree of freedom per cusp torus). This is in fact what happens. The proof is not straightforward. One place where it is written down is the paper *Positively oriented ideal triangulations on hyperbolic three-manifolds* by Young-Eun Choi: `https://www.sciencedirect.com/sdfe/reader/pii/S0040938304000151/pdf`

If you finish the above exercises with time to spare, here are two more good problems to think about.

6. Do Exercise 3.4 in Purcell's book, proving that the developing map is well-defined and only depends on the homotopy class of a path.

7. Working out the dimension of the space of solutions is somewhat easier in dimension 2 (shearing parameters) than in dimension 3 (edge gluing equations).

Let $S = S_{g,n}$ be a surface with $n > 0$ punctures, and with $\chi(S) < 0$. Let $\mathcal{T}$ be an ideal triangulation of S. Check that $\mathcal{T}$ has exactly $|3\chi(S)| = 6g + 3n - 6$ edges.

For each puncture v of S, there is a shearing equation (the sum of shears about v must equal 0). Prove that the n equations are all linearly independent. Since all the equations are linear, it follows that the solution set (a.k.a. the Teichmüller space of S) is $\mathbb{R}^{6g+2n-6}$.

Hint: If you are allowed to choose your favorite triangulation of $S = S_{g,n}$, this statement can be proved by induction on n (relate an ideal triangulation of $S_{g,n}$ to an ideal triangulation of $S_{g,n-1}$). Once you do this, it remains to show that the dimension of the space of solutions does not change when you change the triangulation.