Math 9120 Homework 1

Due Thursday, 9/15/16

1. Let $\alpha \subset T^2$ be a simple closed curve on the torus. Prove that α represents a primitive element of $\pi_1(T^2)$.

2. Let α, β be closed curves on any (orientable) surface S, intersecting transversely if at all. Prove that the algebraic intersection number $\hat{i}(\alpha, \beta)$ only depends on the homology classes of α and β.

3. Let $a, b \in \pi_1(T^2) = \mathbb{Z}^2$. Write $a = \begin{bmatrix} p \\ q \end{bmatrix}$ and $b = \begin{bmatrix} r \\ s \end{bmatrix}$. Prove that

$$\hat{i}(a,b) = \det \begin{bmatrix} p & r \\ q & s \end{bmatrix} \qquad \text{and} \qquad i(a,b) = \left| \det \begin{bmatrix} p & r \\ q & s \end{bmatrix} \right|.$$

Choose your own adventure for the next problem.

4 (easier). Let $P \subset \mathbb{H}^2$ be a polygon with interior angles $\theta_1, \ldots, \theta_n$. Then we have an area formula

$$\operatorname{area}(P) = \sum_{i=1}^{n} (\pi - \theta_i) \; - 2\pi.$$

Use the area formula to show that for a hyperbolic surface S,

$$\operatorname{area}(S) = -2\pi\chi(S).$$

For full credit, you should treat surfaces with punctures and/or boundary.

4 (harder). Let S be a hyperbolic surface. Corollary 1.9 of the book shows that for non-trivial *simple* closed curves α, β, in homotopy classes a, b, the geometric intersection number $i(a, b)$ is realized by geodesics. Prove the same statement for non-simple closed curves. In particular, this requires a careful discussion of how to count intersection numbers when a geodesic happens to self-intersect in points other than double points. You will also want to generalize the bigon criterion.