Problem List

Math 9071, Hyperbolic Geometry, Spring 2026

The date next to each problem indicates the last class day whose material would be needed to solve it.

1. (Jan 12) Let (M, g) be a Riemannian manifold. The Riemannian metric g defines a distance function $d\colon M \times M \to \mathbb{R}$, where $d(p, q)$ is the infimal length of any piecewise smooth curve from p to q. Prove that d is indeed a metric on M, and that the topology induced by this metric agrees with the manifold topology on M.

2. (Jan 14) Let D^n be the disk model of hyperbolic space. Prove by a direct computation that inversion in a sphere perpendicular to ∂D^n is an isometry of the hyperbolic metric.

3. (Jan 14) Prove by induction on n that the orthogonal group $O(n)$ is generated by reflections. In fact, any element of $O(n)$ can be expressed as the product of at most n reflections. Conclude that every isometry of $\mathbb{H}^n$ is the product of a most $n+1$ reflections.

4. (Jan 14) Let $C \subset \mathbb{H}^2$ be a circle in the hyperbolic metric (that is, the set of points at fixed hyperbolic distance from its center). Prove that C also appears as a circle in the upper half-space model H^2. How does one identify the hyperbolic center of C in the upper half-space model?

5. (Jan 14) For any triple of positive angles α, β, γ such that $\alpha + \beta + \gamma < \pi$, prove that there exists a triangle in $\mathbb{H}^2$ with geodesic sides, and with those three angles at the corners. Then, prove that this triangle is unique up to isometry. *Note:* This gives an "angle-angle-angle" type theorem, which does not exist in Euclidean geometry.

6. (Jan 21) Let x, y, z be a three points of $\mathbb{H}^2$ that do not lie on a common line. Prove that a fourth point w is completely determined by its distances to x, y, z. (*Hint:* think about intersections of circles.) Conclude that an isometry of $\mathbb{H}^2$ is completely determined by where it sends x, y, z.

   The above statement has the following generalization: an isometry of $\mathbb{H}^n$ is completely determined by where it sends any "generic" $(n+1)$ tuple of points, where "generic" means that the points do not lie on a lower-dimensional subspace $\mathbb{H}^d \subset \mathbb{H}^n$.

7. (Jan 28) Let φ be an elliptic isometry of $\mathbb{H}^n$. Prove that the set of points fixed by φ is a subspace $\mathbb{H}^m \subset \mathbb{H}^n$, for some $m \leq n$. *Hint:* This is easiest to see in the disk model D^n, but it is also satisfying to give a "synthetic" proof based on the fact that two points determine a geodesic, three non-collinear points determine a copy of $\mathbb{H}^2$, etc.

8. (Jan 28) Prove that the isometry group $\mathrm{Isom}(\mathbb{H}^n)$ acts transitively on triples of points in $\partial\mathbb{H}^n$. *Hint:* Start with $\mathbb{H}^2$, and then bootstrap.