Math 9071 Homework 1

Due Thursday, 3/2/17

1. Let $f : M^n \to N^n$ be a map of degree $d \neq 0$. Prove that the index $[\pi_1 N : f_*(\pi_1 M)]$ is finite and divides d.

2. Let S_g and S_h be surfaces of genus g and h, where $g < h$. Prove that every map $f : S_g \to S_h$ has degree 0. *Hint:* This can be done using problem 1. It can also be done using the Gromov norm (if you know what that is).

3. Prove that the space of marked right-angled hexagons is homeomorphic to $\mathbb{R}^3_+$, parametrized by the lengths of three non-adjacent sides. *Hint:* Thurston's book outlines this in 3 different ways. Any one of them works. See Problem 2.4.11, Problem 2.6.9, or Figure 4.15.

4. Let $S = S_{g,n}$ be a surface with $n > 0$ punctures but no boundary, and with $\chi(S) < 0$. The following outline gives an alternate (non-Fenchel-Nielsen) proof that $\mathcal{T}(S) \cong \mathbb{R}^{6g+2n-6}$. Thurston's book is a good reference for this problem.

(a) Let T be an ideal triangulation of S. Check that T has exactly $|3\chi(S)| = 6g + 3n - 6$ edges.

(b) In every hyperbolic metric on S, each edge of T, define a *shearing parameter*. This is defined as follows. Inscribe a circle in each triangle. Then, for an edge $e \subset T$, the shear $s(e)$ is the signed distance between the points of tangency of the two circles on the two sides of e. The sign is defined to be positive if the circle to the right of e is below the circle to the left of e. Check that this is well-defined (no need to write anything for this part).

(c) Check that a hyperbolic metric on S is complete if and only if the following condition holds: for every puncture of S, the sum of the shears of the edges into that puncture is 0. *Hint:* Map a puncture p to ∞ in the upper half-plane model, and draw the sequence of triangles that meet at that puncture.

(d) The completeness equations introduce exactly n linear constraints into a system with $6g + 3n - 6$ variables. Thus $\mathcal{T}(S) \cong \mathbb{R}^{6g+2n-6}$.