Math 9023 Homework 5

Due Wednesday, 10/31/17

1. Let $S = S_{5,0}$ be a closed surface of genus 5. Find elements of $MCG(S)$ of order 4, 5, and 6. Draw pictures of the corresponding homeomorphisms.

2. Let α, β be simple closed curves on S such that $i(\alpha, \beta) = 1$.

a) For any homeomorphism h, check that $T_{h(\alpha)} = hT_\alpha h^{-1}$.

b) Prove that $(T_\alpha T_\beta)T_\alpha(T_\alpha T_\beta)^{-1} = T_\beta$. Observe that the last equation is equivalent to the "braid relation" $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$.

3. Let S be the torus. Observe that $H_1(S) \cong \mathbb{Z}^2$, hence $\operatorname{Aut} H_1(S) \cong GL(2, \mathbb{Z})$. Any self-homeomorphism $f : S \to S$ induces an automorphism of $H_1(S)$, which we denote $f_* \in GL(2, \mathbb{Z})$. Since f_* only depends on the homotopy class of f, we get a well-defined function $\Psi : MCG(S) \to GL(2, \mathbb{Z})$, namely $\Psi([f]) = f_*$.

a) When f is orientation–preserving, show that f_* has determinant 1, hence $\Psi([f]) \in SL(2, \mathbb{Z})$.

b) Show that $\Psi(MCG(S)) = SL(2, \mathbb{Z})$.

Remark: Since the universal cover of S is contractible, it follows from algebraic topology that Ψ is 1-to-1. Compare Proposition 1B.9 of Hatcher. As a consequence, $MCG(S) \cong SL(2, \mathbb{Z})$.

4. Consider the cell complex X obtained by gluing two tetrahedra along their faces, as specified by the picture below. Prove that X has one vertex v, and that the link of this vertex is a torus.

Note: The figure is from Thurston's *Three-Dimensional Geometry and Topology*, and is significant because $X \setminus v$ is homeomorphic to the complement of the figure 8 knot in S^3.

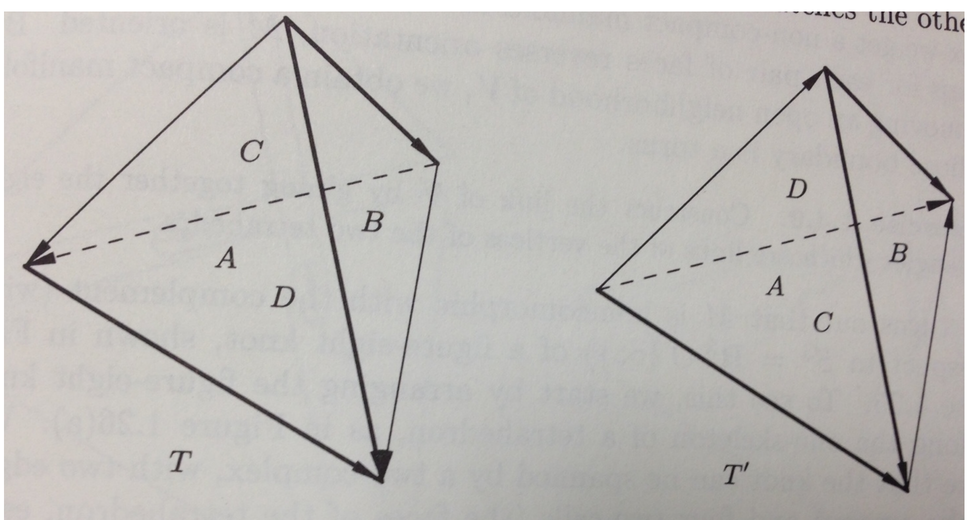

Figure 1.24. A simple three-dimensional gluing pattern. A simple pattern for gluing two tetrahedra. Each face has a label (centered on it), and faces with the same label are identified in a way that is unambiguously determined by the requirement that edge types (thick and thin) and directions match. (In fact, even the pairing of faces could be reconstructed from this requirement.)

5. Let $K \subset S^3$ be a knot, and let R be the 3–manifold obtained by (p, q) Dehn surgery on K. That is, the meridian of the solid torus is glued to the curve $p\mu + q\lambda$ in $M = S^3 \setminus N(K)$.

a) Use the Mayer–Vietoris sequence to show that $H_1(R) \cong \mathbb{Z}/p$.

b) Conclude that not every 3–manifold can be obtained by Dehn surgery on a knot.