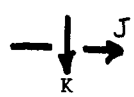
D. LINKING NUMBERS.

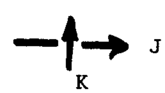
Let J and K be two disjoint oriented knots in S^3 (or R^3). This section describes eight methods for defining an integer called their <u>linking number</u>, all of which turn out to be equivalent, at least up to sign. Assume J and K are polygonal.

(1) Let $[J]$ be the homology class in $H_1(S^3 - K)$ carried by J. Since $H_1(S^3 - K) \cong Z$, we may choose a generator γ of this group and write $[J] = n\gamma$. Define $\ell k_1(J,K) = n$.

(2) Let M be a PL Seifert surfact for K, with bicollar (N,N^+,N^-) of $\overset{\circ}{M}$ as in the previous section. Assume (allowing adjustment of J by a homotopy in $S^3 - K$) that J meets M in a finite number of points, and at each such point J passes locally (a) from N^- to N^+ or (b) from N^+ to N^-, following its orientation. Weight the intersections of type (a) with $+1$ and those of type (b) with -1. The sum of these numbers we denote by $\ell k_2(J,K)$. [Note that this seems to depend on M].

(3) Consider a regular projection of $J \cup K$. At each point at which J crosses <u>under</u> K, count

$+1$ for 

and -1 for 

The sum of these, over all crossings of J under K, is called $\ell k_3(J,K)$.

(4) J is a loop in $S^3 - K$, hence represents an element of $\pi_1(S^3 - K)$ with suitable basepoint. This fundamental group abelianizes to Z, and the loop J is thereby carried to an integer, called $\ell k_4(J,K)$.

(5) [J] and [K] are 1-cycles in S^3. Choose a 2-chain $C \in C_2(S^3;Z)$ such that $[K] = \partial C$. Then the intersection $C\cdot[J]$ is a 0-cycle, well-defined up to homology. Since $H_o(S^3) \cong Z$, $C\cdot[J]$ corresponds to an integer which we call $\ell k_5(J,K)$.

(6) Regard $J, K : S^1 \to R^3$ as maps.

In vector notation, define a map $f : S^1 \times S^1 \to S^2$ by the formula

$$f(u,v) = \frac{K(u) - J(v)}{|K(u) - J(v)|} .$$

If we orient $S^1 \times S^1$ and S^2 then f has a well-defined degree. Let $\ell k_6(J,K) = \deg(f)$.

(7) (Gauss Integral) Define $\ell k_7(J,K)$ to be the integer

$$\frac{1}{4\pi}\int_J\int_K \frac{(x'-x)(dydz'-dzdy') + (y'-y)(dzdx'-dxdz') + (z'-z)(dxdy'-dydx')}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}}$$

where (x,y,z) ranges over J and (x',y',z') over K.

(8) Let $p : \tilde{X} \to X$ be the infinite cyclic cover of $X = S^3 - K$ and let τ generate $\mathrm{Aut}(\tilde{X})$. Consider J as a loop in X based at, say, $x \in \mathrm{Im}\, J$. Lift J to a path $\tilde{J}$ in $\tilde{X}$, starting at any $\tilde{x}_o \in p^{-1}(x)$ and call its terminal point $\tilde{x}_1 \in p^{-1}(x)$. There is a unique integer m such that $\tau^m(\tilde{x}_o) = \tilde{x}_1$. Define $\ell k_8(J,K) = m$.

1. EXERCISE. Identify the choice in each of the above definitions which affects the sign of the linking number.

2. THEOREM. $\ell k_i = \pm \ell k_j$; $i, j = 1, \ldots, 8$.

PROOF: $\ell k_1 = \pm \ell k_4$: since the Hurewicz homomorphism $h : \pi_1(S^3-K) \to H_1(S^3-K)$ which carries loops to 1-cycles is just the abelianization map.

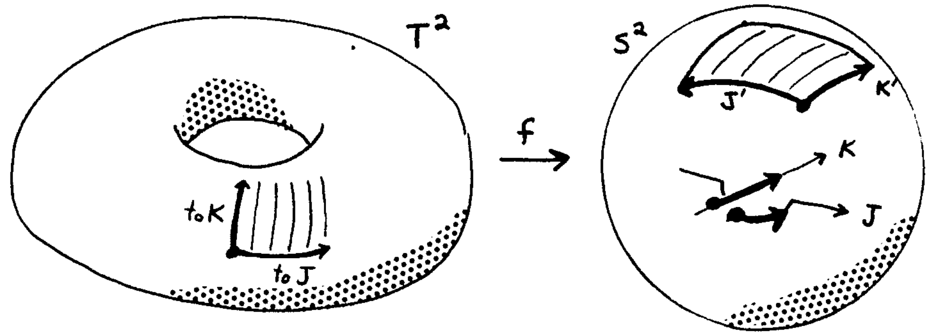
$\ell k_2 = \pm \ell k_5$: since we may take the C of (5) to be the 2-cycle carried by M.

$\ell k_2 = \pm \ell k_3$: Using the given regular projection of (3) construct a Seifert surface M for K according to the proof of theorem A4, so that J is above M except near the underpasses and intersects M once at each underpass. If the disks are bicollared in such a way that K runs counterclockwise around the boundary as viewed from above, then ± 1 is assigned to the underpasses in the same way in (2) as in (3).

$\ell k_4 = \pm \ell k_8$: As described in Appendix A, the τ^m of (8) is just the automorphism τ_J induced by the loop J and the equality follows from the isomorphism $\mathrm{Aut}(\tilde{X}) \cong \pi_1(X)$ / (commutator subgroup).

$\ell k_2 = \pm \ell k_8$: Construct $\tilde{X}$ using the M of (2) by the method of the previous section. Choose $\tilde{x}_0 \in Y_0 \subset \tilde{X}$. Then each intersection of type (a) corresponds to a passage of $\tilde{J}$ from Y_i to Y_{i+1}, while type (b) intersections to the reverse. So $\tilde{J}$ ends up in Y_r, $r = \ell k_2(J,K)$. But if $\tau : \tilde{X} \to \tilde{X}$ is chosen as the shift $Y_i \to Y_{i+1}$ we conclude that $\ell k_8(J,K) = \ell k_2(J,K)$.

$\ell k_3 = \pm \ell k_6$: suppose there is a point $z \in S^2$ such that $f^{-1}(z)$ is a finite set and f is a homeomorphism near each point of $f^{-1}(z)$. Then $\deg f$ may be calculated by adding the points, weighted -1 if f locally reverses orientation and $+1$ if f locally preserves orientation. But there is such a point, namely the point $z \in S^2$ directly above the projection plane of (3), corresponding to the viewer's eye: $f^{-1}(z)$ has one element for each crossing of J under K. The picture below shows why the two types of crossings correspond to different orientations.



$\ell k_7 = \pm \ell k_6$: see Spivak's Calculus on Manifolds.

This integral (or its negative) is just an analytic way of computing $\deg f$.

3. DEFINITION. Define the <u>linking number</u> $\ell k(J,K)$ to be any of the above.

4. REMARK. The sign ambiguity is usually not a bother, and disappears if one chooses a 'convention' as in (3). Note that definition (6) (and which others?) does not require that J and K be embeddings of S^1, as long as they are disjoint, so the notion of linking number extends to arbitrary disjoint curves in S^3 or R^3 .

5. THEOREM: If there are homotopies $J_t : S^1 \to R^3$ and $K_t : S^1 \to R^3$ so that $\mathrm{Im}\, J_t$ is disjoint from $\mathrm{Im}\, K_t$ for each $0 \le t \le 1$, then $\ell k(J_o,K_o) = \ell k(J_1,K_1)$.

PROOF. Using (6) define $f_t(u,v) = \dfrac{K_t(u,v) - J_t(u,v)}{|K_t(u,v) - J_t(u,v)|}$ and we have homotopic maps $f_o, f_1 : S^1 \times S^1 \to S^2$. Hence they have the same degree.

6. THEOREM:

$$\ell k(J,K) = \ell k(K,J)$$

$$\ell k(-J,K) = -\ell k(J,K)$$

where $-J$ is J with the reverse orientation.