Math 9023 Homework 5

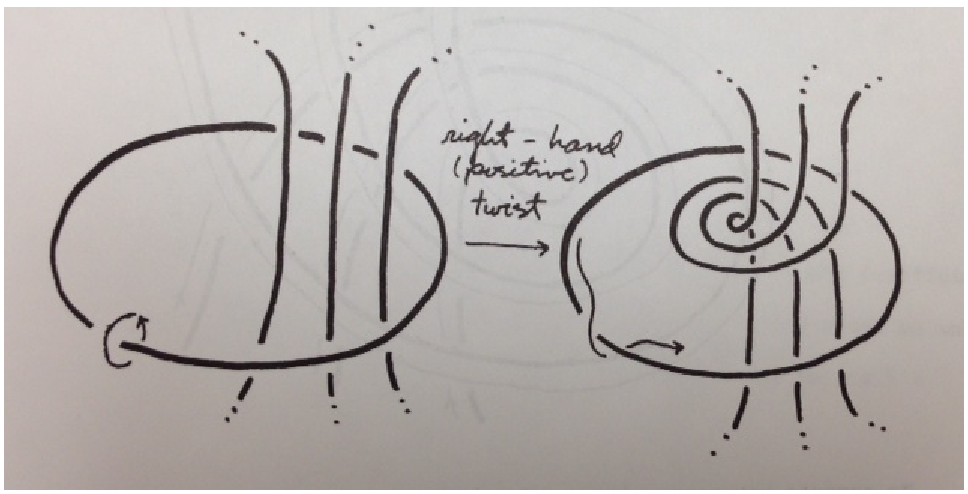
Due Thursday, 11/20/14

1. Let M be a closed, even-dimensional manifold that is the boundary of W. Prove that $\chi(M) = 2\chi(W)$. (Note that this finishes the proof that every manifold that bounds has even Euler characteristic.)

2. Suppose that a closed 3–manifold M is obtained by integer Dehn surgery along a link $L \subset S^3$. Modify L by performing a full twist along a disk bounded by an unknot $K \subset L$, as shown in the figure below. Prove that after this modification, the surgery coordinate $r_i \in \mathbb{Z}$ along some component $K_i \neq K$ becomes

$$r_i' = r_i + lk(K_i, K)^2.$$

Hint: If you get stuck, see Figure 16.9(a) in Prasolov & Sossinsky.



3. On page 109, Prasolov & Sossinsky compute that +1 surgery on the trefoil produces a 3–manifold P, called the Poincaré dodecahedral sphere. (There is a small mistake: the trefoil in Figure 18.5 has its crossings reversed.) As they compute,

$$I^* = \pi_1(P) = \langle x, y : xyx = yxy, yx^2y = x^3 \rangle.$$

a) Setting $a = x$, $b = xy$, prove that

$$I^* \cong \langle a, b : a^5 = b^3 = (ba)^2 \rangle.$$

b) Prove that I^* surjects onto A_5, the alternating subgroup of S_5.

c) Prove that I^* surjects onto the group of orientation–preserving symmetries of a regular dodecahedron. (This group is known to be isomorphic to A_5, but construct the surjection directly. It's very pretty.)

d) For extra credit, show that the kernel of either surjection is the center of I^*, which has order 2. Therefore, I^* is a twofold extension of A_5, and has order 120. (Warning: this is hard.)

4. Let M be the 3–manifold obtained by -1 surgery on the trefoil.

a) Following Problem 18.3, prove that $\pi_1(M) \cong \langle a, b : a^7 = b^3 = (ba)^2 \rangle$.

b) Find (without proof) a free group on 2 generators in $\pi_1(M)$. It follows that M and P are not homeomorphic, hence $+1$ and -1 surgeries on the trefoil produce different 3–manifolds.

c) For extra credit, prove that your answer in (b) is indeed a free group. The following page may be useful: `http://en.wikipedia.org/wiki/Triangle_group`