Homework 3

Due Wednesday, 1/31/07

1. Let T be a torus with one point removed. Compute $\pi_1(T)$. *Hint: there is a deformation retraction from T onto a space that you already understand.*

2. Let M be a (path–connected) manifold of dimension n. Remove an open ball B^n from M. Prove that when $n \geq 3$,

$$\pi_1(M \setminus B^n) \cong \pi_1(M).$$

(Note that by the previous problem, the assumption $n \geq 3$ is necessary.)

3. There is a standard way to glue together two manifolds M and N of the same dimension. Remove an open ball B^n from each of M and N, and glue $M \setminus B^n$ to $N \setminus B^n$ along the two $(n-1)$ dimensional boundary spheres. The resulting manifold is called the *connected sum* of M and N, and is denoted $M \# N$. Prove that when $n \geq 3$,

$$\pi_1(M \# N) \cong \pi_1(M) * \pi_1(N).$$

4. Let Γ be a connected graph. Let E be an edge of Γ that is not a loop (the endpoints of E are distinct). Let $\Delta = \Gamma/E$, the graph obtained by identifying E to a point. Prove that the quotient map $\varphi : \Gamma \to \Delta$ is a homotopy equivalence. (This follows from Proposition 0.17 in Chapter 0, but in this case you should prove it directly.)

5. Let Γ be a connected graph that has v vertices and e edges. Prove that $\pi_1(\Gamma)$ is the free group on $(e - v + 1)$ generators. *Hint: use Problem 4.*