Math 8161 Homework 7

Due Tuesday, 12/8/09

1. Let $X = S^2 \cup A$, where A is an axis connecting the north and south poles of S^2. Describe the universal cover $\widetilde{X}$ of X and the action of $\pi_1(X)$ on the universal cover.

2. Problem 2 on the last homework implies that the manifold $\mathbb{RP}^3 \# \mathbb{RP}^3$ has fundamental group

$$\pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3) \cong \mathbb{Z}/2 * \mathbb{Z}/2 \cong \langle a, b : a^2 = b^2 = 1 \rangle.$$

This group contains an index–2 subgroup $H = \{(ab)^n\} \cong \mathbb{Z}$. Prove that the double cover corresponding to this subgroup is $S^2 \times S^1$, by constructing an explicit covering map.

3. Let X be the figure–8 graph with one vertex and two edges. Let H be the subgroup of $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$ generated by the cubes of all elements. Construct the covering map $p : Y \to X$ corresponding to H, and describe the action of the deck transformation group on the cover. *Hint:* this covering space Y has 27 sheets, and can be drawn on a torus so that the complementary regions are nine triangles labeled aaa, nine triangles labeled bbb, and nine hexagons labeled $ababab$.

4. Let X be the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus. Compute $\pi_1(X)$, describe the universal cover $\widetilde{X}$ of X, and describe the action of $\pi_1(X)$ on $\widetilde{X}$.

5. *Extra credit.* A covering map $p : Y \to X$ is called *regular* if the corresponding subgroup $p_* \pi_1(Y, y_0)$ is normal in $\pi_1(X, x_0)$. Otherwise, the cover is *irregular.* Construct irregular covers of the Klein bottle by the Klein bottle and by the torus.