Math 8161 Homework 7

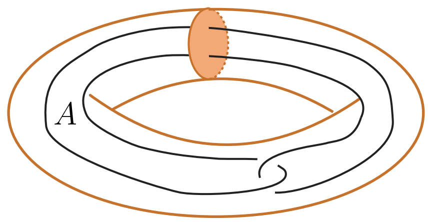
Due Tuesday, 11/24/09

1. Prove that there are no retractions $r : X \to A$ for each of the following cases:

a) $X = \mathbb{R}^3$ and A is any knot (embedded circle) in $\mathbb{R}^3$.

b) $X = S^1 \times D^2$ and A is its boundary torus $S^1 \times S^1$.

c) $X = S^1 \times D^2$ and A is the circle shown in the figure below.



2. There is a standard way to glue together two (path–connected) manifolds M and N of the same dimension. Remove an open ball B^n from each of M and N, and glue $M \smallsetminus B^n$ to $N \smallsetminus B^n$ along the two $(n-1)$ dimensional boundary spheres. The resulting manifold is called the *connected sum* of M and N, and is denoted $M \# N$.

a) Prove that when $n \geq 3$, $\pi_1(M \smallsetminus B^n) \cong \pi_1(M)$.

b) Prove that when $n \geq 3$, $\pi_1(M \# N) \cong \pi_1(M) * \pi_1(N)$.

3. Let Γ be a connected graph that has v vertices and e edges.

a) Let E be an edge of Γ that is not a loop (the endpoints of E are distinct). Let $\Delta = \Gamma/E$, the graph obtained by identifying E to a point. Prove that the quotient map $\varphi : \Gamma \to \Delta$ is a homotopy equivalence.

b) Prove that $\pi_1(\Gamma)$ is the free group on $(e - v + 1)$ generators.

4. Let $X = S^2 \cup A$, where A is an axis connecting the north and south poles of S^2. Find a cell complex structure on X, and use it to compute the fundamental group.

5. Let C be the unit cube in $\mathbb{R}^3$, and let M be the manifold obtained by identifying opposite faces of C with a $90°$ clockwise twist. (This is analogous to the Poincaré dodecahedral space, but with a cube instead of a dodecahedron.) Find a cell complex structure on M, and use it to show that $\pi_1(M)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.