Math 8161 Homework 6

Due Thursday, 11/12/09

1. Let X be a metric space, with the property that every point $x \in X$ has a metric neighborhood $B_\epsilon(x)$ with compact closure. Let G be a group that acts freely and properly on X. Prove that the quotient map $p : X \to X/G$ is a covering map.

2. Let $p : Y \to X$ be a covering map, and assume that X is connected. Show that if $p^{-1}(x_0)$ has k elements for some $x_0 \in X$, then $p^{-1}(x)$ will have k elements for every $x \in X$. In this case, Y is called a *k–fold cover* of X.

3. Let X be a simply connected topological space, and $p : Y \to X$ a covering map. If Y is path-connected, show that p is a homeomorphism.

4. Given a topological space X, prove that the following are equivalent:

a) Every map $f : S^1 \to X$ is homotopic to a constant map, with image a single point.

b) Every map $f : S^1 \to X$ extends to a map $f : D^2 \to X$.

c) $\pi_1(X, x_0) = \{1\}$ for every $x_0 \in X$.

5. A subset $A \subset \mathbb{R}^n$ is called *star shaped* if for some point $a_0 \in A$, all the line segments joining a_0 to other points of A lie in A.

a) Find an example of a star-shaped set that is not convex.

b) Show that if A is star-shaped, then it is simply connected.

6. Let f and g be loops in the product space $X \times Y$, both based at (x_0, y_0). Suppose that the image of f is contained in $X \times \{y_0\}$ and the image of g is contained in $\{x_0\} \times Y$. Prove by a direct argument that f and g represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. That is, construct a homotopy from $f \cdot g$ to $g \cdot f$.