Math 8161 Homework 5

Due Thursday, 10/29/09

1. Let M be a manifold. Prove that the connected components of M are exactly the same as its path components, and that each of those components is itself a manifold.

2. Let $SL(n,\mathbb{R})$ be the set of $n \times n$ matrices with determinant 1, considered as a subspace of $\mathbb{R}^{n^2}$.

a) Prove that $SL(n,\mathbb{R})$ is a manifold of dimension $n^2 - 1$. *Hint:* Think about the degrees of freedom in choosing the image of a standard basis for $\mathbb{R}^n$.

b) Is $SL(n,\mathbb{R})$ compact?

c) How many components does it have?

3. Let $O(3)$ be the set of 3×3 matrices A, such that $\|Av\| = \|v\|$ for every vector $v \in \mathbb{R}^3$.

a) Prove that $O(3)$ is a manifold. What is its dimension?

b) Is $O(3)$ compact?

c) How many components does it have?

4. Let P be the Poincaré docecahedral space, obtained by gluing opposite faces of a dodecahedron with a 1/10 clockwise twist. (See the figure on page 88 or our book, or feel free to play with the model in my office.) How many vertices does P have after the gluing? How many edges, faces, and 3–dimensional cells? Use this to compute the Euler characteristic of P.

5. Let $X = \mathbb{R}^2$, and let G be the group action generated by the following two elements:

$$(x, y) \mapsto (x, y + 1), \qquad (x, y) \mapsto (x + 1, -y).$$

What is the quotient space X/G? This is an object you have seen before.

6. Let $X = \mathbb{R}^2$, tiled by regular hexagons. Let G be the group of all translations of $\mathbb{R}^2$ that preserve the tiling (sending each hexagon onto another hexagon).

a) What group is G, algebraically?

b) What is the quotient X/G?