Math 8161 Homework 4

Due Thursday, 10/15/09

1. Let $\{A_i\}$ be a family of connected subsets of a topological space X, such that $A_i \cap A_j \neq \emptyset$ for all i, j. Show that $\cup_i A_i$ is connected.

2. Let $A \subset X$ be a connected set, and suppose that $A \subset Y \subset \overline{A}$. Prove that Y is connected.

3. Is a product of path-connected spaces necessarily path-connected?

4. Let $X = \mathbb{R}^{\mathbb{N}}$ be the set of all sequences of real numbers. Thus a point of X has the form $x = (x_1, x_2, x_3, \ldots)$. Define a metric D on X by

$$D(x,y) \;=\; \begin{cases} 1, & \text{if } |x_n - y_n| \geq 1 \text{ for some } n \in \mathbb{N}, \\ \sup\{|x_n - y_n| : n \in \mathbb{N}\}, & \text{otherwise.} \end{cases}$$

Prove that x and y lie in the same path component of $\mathbb{R}^{\mathbb{N}}$ if and only if the sequence

$$x - y \;=\; (x_1 - y_1, x_2 - y_2, \ldots)$$

is bounded.

5. For each of the following spaces, find an atlas of finitely many charts to $\mathbb{R}^n$ (with the appropriate n).

a) The circle S^1.

b) The torus $T^2 = S^1 \times S^1$.

c) The projective plane $\mathbb{RP}^2$, obtained as the quotient of S^2 with antipodal points identified.

d) The 3–torus $T^3 = S^1 \times S^1 \times S^1$.

6. *Extra credit.* Let X be the set of all lines in $\mathbb{R}^2$, with the topology (or the metric) coming from the charts that we discussed in class. Let $Y \subset X$ be the set of all lines whose Euclidean distance to the origin is at most 1. Prove that Y is homeomorphic to a Möbius strip.

Hint: the correspondence between lines and polar coordinates for points is key here.