Math 8161 Homework 1

Due Thursday, 9/10/09

1. Suppose (X, D) is a metric space. Which of the following functions are metrics on X?

a) $D_1(x, y) = k\, D(x, y)$, where k is a positive real number.

b) $D_2(x, y) = k\, D(x, y)$, where k is any real number.

c) $D_3(x, y) = D^n(x, y)$, where n is a positive integer.

d) $D_4(x, y) = D^r(x, y)$, where $0 < r < 1$.

2. Let P and P' be points of $\mathbb{R}^2$, and let

$$M(P, P') = \{Q \in \mathbb{R}^2 : d(P, Q) = d(P', Q)\},$$

where d is some metric on $\mathbb{R}^2$. Describe geometrically what $M(P, P')$ will look like when d is the Euclidean metric, the taxicab metric, and the max metric.

3. Let (X, D) be a metric space. Let $\{F_i\}$, $i \in I$ be a family of closed sets with the following property: for every $x \in X$, there is a $r > 0$, such that $B_r(x)$ intersects only finitely many of the F_i. Prove that the union $\cup_I F_i$ is closed.

4. Prove that the Euclidean, taxicab, and max metrics on $\mathbb{R}^2$ are *equivalent* – that is, they have the same open sets. *Note:* rephrasing this in terms of continuity, or in terms of sequences, might make the checking less tedious.

5. Let $B[0, 1] = \{f : [0, 1] \to \mathbb{R} : f \text{ is bounded}\}$ be the set of all bounded functions on $[0, 1]$, equipped with the usual sup metric:

$$D(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}.$$

Define a function $\varphi : B[0, 1] \to \mathbb{R}$ by the formula $\varphi(f) = f(0)$. Is φ continuous? Prove your answer.

6. Let (X, D) be a metric space, and $S \subset X$ an arbitrary subset. For any point $x \in X$, the distance to S is defined to be

$$D(x, S) = \inf\{D(x, y) : y \in S\}.$$

Now, we can construct a function $f : X \to \mathbb{R}$ by $f(x) = D(x, S)$. Prove that f is continuous.