Math 8062 Homework 5

Due Thursday, 2/24/22

1. Problem 2 of the third homework implies that the manifold $\mathbb{RP}^3 \# \mathbb{RP}^3$ has fundamental group

$$\pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3) \;\cong\; \mathbb{Z}/2 * \mathbb{Z}/2 \;\cong\; \langle a, b : a^2 = b^2 = 1 \rangle.$$

This group contains an index–2 subgroup $H = \{(ab)^n\} \cong \mathbb{Z}$. Construct a double cover $p : S^2 \times S^1 \to \mathbb{RP}^3 \# \mathbb{RP}^3$ such that the induced subgroup $p_* \pi_1(S^2 \times S^1)$ is exactly H.

2. Let X be the cell complex obtained by identifying opposite faces of a cube with a 90° twist (as in problem 14 on page 54). Prove that every map $f : X \to S^1$ is homotopic to a constant map. *Hint:* use the lifting criterion.

3. Do problem 4 on page 79 of Hatcher.

4. Let X be the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

a) Construct a double cover $Y \to X$, where $Y \cong \Gamma \times \Delta$ is the product of two graphs.

b) Describe the universal cover $\widetilde{Y} = \widetilde{X}$. *Hint:* consider problem 2 on page 79.

c) Describe the action of $\pi_1(X)$ on $\widetilde{X} = \widetilde{Y}$.