Math 8062 Homework 4

Due Thursday, 2/17/22

The problems on this homework outline a proof of the following result, mentioned in class.

Theorem 1 *For every finitely presented group $G = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_k \rangle$, there is a compact orientable 4–manifold X such that $\pi_1(X) \cong G$.*

Start with $V = \#_n(S^3 \times S^1)$. It follows from the last homework that $\pi_1(V) \cong \langle g_1, \ldots, g_n \rangle$. Every relator r_i is represented by a loop based at $x_0 \in V$. Let $\varphi_i : S^1 \to V$ be a (smooth) loop in V, in the same free homotopy class as r_i.

1. Show that the smooth loops $\varphi_1, \ldots, \varphi_k$ can be modified via free homotopy until they are embeddings with disjoint images. (This step only requires the dimension of V to be at least 3.)

For every i, let N_i be an open ϵ–neighborhood of the image $\varphi_i(S^1)$. Then $N_i \cong S^1 \times B^3$. We choose ϵ small enough that the neighborhoods are disjoint. Let $W = V \setminus (N_1 \cup \ldots \cup N_k)$.

2. Show that the inclusion $W \hookrightarrow V$ induces an isomorphism of fundamental groups.
Hint: Consider the example of lens spaces from class. What are the dimensions of cells that need to be added to W in order to recover V?

By construction, ∂W consists of k copies of $(S^1 \times S^2)$. Let X be the 4–manifold obtained by attaching a copy of $(D^2 \times S^2)$ to every component of ∂W.

3. Show that the inclusion $W \hookrightarrow X$ induces a surjection of fundamental groups, whose kernel is exactly the normal subgroup $\langle\langle r_1, \ldots, r_k \rangle\rangle$. In particular, explain why the free homotopy of φ_i performed in Problem 1 does not cause any problems in the computation of $\pi_1(X)$.

This concludes the proof of the theorem that $\pi_1(X) \cong \langle g_1, \ldots, g_n \mid r_i, \ldots, r_k \rangle$. □

4. The above proof has a straightforward generalization to every dimension $d \geq 4$. Give a quick description of how the dimensions of the above-mentioned objects should be modified in order for the argument to work in dimension d.