Math 8062 Homework 4

Due Thursday, 2/24/11

1. Let $X = S^2 \cup A$, where A is an axis connecting the north and south poles of S^2. Describe the universal cover $\widetilde{X}$ of X and the action of $\pi_1(X)$ on the universal cover.

2. A covering map $p : Y \to X$ is called *regular* if the corresponding subgroup $p_* \pi_1(Y, y_0)$ is normal in $\pi_1(X, x_0)$. Otherwise, the cover is *irregular*. Construct irregular covers of the Klein bottle by the Klein bottle and by the torus.

3. Let M and N be (topological) manifolds, with a covering map $p : N \to M$. Let $\mathcal{A} = \{(U_\alpha,\, \varphi_\alpha)\}$ be a smooth atlas on M, such that $\varphi_\alpha(U_\alpha)$ is an open ball $B_\alpha \subset \mathbb{R}^n$.

a) For each $(U_\alpha,\, \varphi_\alpha) \in \mathcal{A}$, prove that $p^{-1}U_\alpha$ is a disjoint union of slices, each mapped homeomorphically to U_α.

b) Prove that

$$\mathcal{B} = \{(V_\alpha,\, \varphi_\alpha \circ p) : V_\alpha \text{ is a component of } p^{-1}U_\alpha\}$$

is a smooth atlas on N.

c) Prove that the smooth atlas $\mathcal{B}$ makes p into a smooth map, and furthermore that $p : V_\alpha \to U_\alpha$ is a diffeomorphism.

d) Let Y be a smooth vector field on M. Construct a smooth vector field Z on N, such that $p_* Z_x = Y_{p(x)}$, for all $x \in N$. (Here, p_* denoted the pushforward of tangent vectors, not the induced action on π_1.) Note that this gives a natural way to *pull back* vector fields to a cover, whereas normally vectors push forward and co-vectors (and differential forms) pull back.