Math 8061 Homework 4

Due Wednesday, 10/4/23

1. Prove that the tangent bundle TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$. *Hint:* Construct a nonzero vector field.

2. Prove that $TS^3 \cong S^3 \times \mathbb{R}^3$, by constructing three smooth vector fields on S^3 that are linearly independent at each point. (*Hint:* S^3 has a group structure, which comes from thinking of it as the set of unit quaternions in $\mathbb{R}^4$. This is useful for constructing vector fields.)

3. Suppose M is a compact manifold and $f : M \to N$ is a 1–1 immersion. Prove that f is a homeomorphism to its image. (This is the definition of an *embedding*.)

4. Let $\mathbb{RP}^2 = S^2/\sim$, where antipodal points on the sphere are identified.

a) Prove that the map $g : \mathbb{RP}^2 \to \mathbb{R}^4$, defined by

$$g([x, y, z]) = (yz,\ xz,\ xy,\ x^2 - y^2),$$

is a smooth 1–1 immersion.

b) Prove that the map $g : \mathbb{RP}^2 \to \mathbb{R}^3$, defined by

$$g([x, y, z]) = (yz,\ xz,\ xy)$$

fails to be an immersion at 6 points (the 6 image points are at distance 1/2 from the origin, on each axis). In fact, there is an immersion $h : \mathbb{RP}^2 \to \mathbb{R}^3$, called Boy's surface. See

`http://en.wikipedia.org/wiki/Boy's_surface`