Math 8061 Homework 1

Due Wednesday, 9/6/23

1. Let M be an n–dimensional manifold with boundary. Prove that the boundary ∂M is an $(n-1)$ dimensional manifold, without boundary.

2. A topological space X is called *connected* if it cannot be expressed as the disjoint union of two non-empty, open sets. X is called *path-connected* if every $x, y \in X$ are joined by a path: that is, there is a continuous map $f : [0,1] \to X$, such that $f(0) = x$ and $f(1) = y$.

a) Prove that every path-connected metric space is connected. You may assume the classical fact that the interval $[0,1]$ is connected.

b) Prove that a connected (topological) manifold is path-connected. *Note:* the book outlines a proof of this in Proposition 1.11, relying on an (unproved) Proposition A.43 in the Appendix. To do this problem, you would need to either prove the result in the appendix, or argue directly.

3. Let M be a topological n–manifold. We say that smooth atlases $\mathcal{A}$ and $\mathcal{B}$ on M are *compatible* if their union $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas. Prove that this gives an equivalence relation on atlases. *Note:* Be careful when checking transitivity of. You will need to unwind several of the definitions.

4. Let $O(3)$ be the set of 3×3 matrices A, such that $\|Av\| = \|v\|$ for every vector $v \in \mathbb{R}^3$.

a) Prove that $O(3)$ is a topological 3–manifold. *Hint:* consider an orthonormal basis $\{v_1, v_2, v_3\}$ and think about the degrees of freedom in choosing the images Av_1, Av_2, Av_3.

b) Is $O(3)$ compact?

c) Is $O(3)$ connected?