Proposition 13.3 (Naturality of the Lie Bracket). *Let $F\colon M \to N$ be a smooth map, and let $V_1, V_2 \in \mathfrak{T}(M)$ and $W_1, W_2 \in \mathfrak{T}(N)$ be vector fields such that V_i is F-related to W_i, $i = 1, 2$. Then $[V_1, V_2]$ is F-related to $[W_1, W_2]$. If F is a diffeomorphism, then $F_*[V_1, V_2] = [F_*V_1, F_*V_2]$.*

Proof. Using Lemma 3.17 and fact that V_i and W_i are F-related,

$$\begin{aligned} V_1V_2(f \circ F) &= V_1((W_2f) \circ F) \\ &= (W_1W_2f) \circ F. \end{aligned}$$

Similarly,

$$V_2V_1(f \circ F) = (W_2W_1f) \circ F.$$

Therefore,

$$\begin{aligned} [V_1, V_2](f \circ F) &= V_1V_2(f \circ F) - V_2V_1(f \circ F) \\ &= (W_1W_2f) \circ F - (W_2W_1f) \circ F \\ &= ([W_1, W_2]f) \circ F. \end{aligned}$$

The result then follows from the lemma. The statement when F is a diffeomorphism is an obvious consquence of the general case, because $W_i = F_*V_i$ in that case. □

Proposition 13.4. *Let N be an immersed submanifold of M, and suppose $V, W \in \mathfrak{T}(M)$. If V and W are tangent to N, then so is $[V, W]$.*

Proof. This is a local question, so we may replace N by an open subset of N that is embedded. Then Proposition 5.8 shows that a vector $X \in T_pM$ is in T_pN if and only if $Xf = 0$ whenever $f \in C^\infty(M)$ vanishes on N. Suppose f is such a function. Then the fact that V and W are tangent to N implies that $Vf|_N = Wf|_N = 0$, and so

$$[V, W]_pf = V_p(Wf) - W_p(Vf) = 0.$$

This shows that $[V, W]_p \in T_pN$, which was to be proved. □

We will see shortly that the Lie bracket $[V, W]$ is equal to the Lie derivative $\mathcal{L}_VW$, even though the two quantities are defined in ways that seem totally unrelated. Before doing so, we need to prove one more result, which is of great importance in its own right. If V is a smooth vector field on M, a point $p \in M$ is said to be a *singular point* for V if $V_p = 0$, and a *regular point* otherwise.

Theorem 13.5 (Canonical Form for a Regular Vector Field). *Let V be a smooth vector field on a smooth manifold M, and let $p \in M$ be a regular point for V. There exist coordinates (x^i) on some neighborhood of p in which V has the coordinate expression $\partial/\partial x^1$.*

Proof. By the way we have defined coordinate vector fields on a manifold, a coordinate chart (U, φ) will satisfy the conclusion of the theorem provided that $(\varphi^{-1})_*(\partial/\partial x^1) = V$, which will be true if and only if φ^{-1} takes lines parallel to the x^1 axis to the integral curves of V. The flow of V is ideally suited to this purpose.

Begin by choosing any coordinates (y^i) on a neighborhood U of p, with p corresponding to 0. By composing with a linear transformation, we may assume that $V_p = \partial/\partial y^1|_p$. Let $\theta\colon \mathcal{D}(V) \to M$ be the flow of V. There exists $\varepsilon > 0$ and a neighborhood $U_0 \subset U$ of p such that the product open set $(-\varepsilon, \varepsilon) \times U_0$ is contained in $\mathcal{D}(V)$ and is mapped by θ into U.

Let $S \subset \mathbb{R}^{n-1}$ be the set

$$S = \{(x^2, \dots, x^n) : (0, x^2, \dots, x^n) \in U_0\},$$

and define a smooth map $\psi\colon (-\varepsilon, \varepsilon) \times S \to U$ by

$$\psi(t, x^2, \dots, x^n) = \theta(t, (0, x^2, \dots, x^n)).$$

Geometrically, for each fixed $(x^2, \dots, x^n)$, ψ maps the interval $(-\varepsilon, \varepsilon) \times \{(x^2, \dots, x^n)\}$ to the integral curve through $(0, x^2, \dots, x^n)$.

First we will show that ψ pushes $\partial/\partial t$ forward to V. We have

$$\begin{aligned} \left(\psi_* \left.\frac{\partial}{\partial t}\right|_{(t_0,x_0)}\right) f &= \left.\frac{\partial}{\partial t}\right|_{(t_0,x_0)} (f \circ \psi) \\ &= \left.\frac{\partial}{\partial t}\right|_{t=t_0} (f(\theta(t, (0, x_0)))) \\ &= V_{\psi(t_0,x_0)} f, \end{aligned} \tag{13.5}$$

where we have used the fact that $t \mapsto \theta(t, (0, x_0))$ is an integral curve of V. On the other hand, when restricted to $\{0\} \times S$, $\psi(0, x^2, \dots, x^n) = \theta(0, (0, x^2, \dots, x^n)) = (0, x^2, \dots, x^n)$, so

$$\psi_* \left.\frac{\partial}{\partial x^i}\right|_{(0,0)} = \left.\frac{\partial}{\partial y^i}\right|_{(0,0)}.$$

Since $\psi_*\colon T_{(0,0)}((-\varepsilon, \varepsilon) \times S) \to T_pM$ takes a basis to a basis, it is an isomorphism. Therefore, by the inverse function theorem, there are neighborhoods W of $(0, 0)$ and Y of p such that $\psi\colon W \to Y$ is a diffeomorphism.

Let $\varphi = \psi^{-1}\colon Y \to W$. Equation (13.5) says precisely that V is equal to the coordinate vector field $\partial/\partial t$ in these coordinates. Renaming t to x^1, this is what we wanted to prove. □

This theorem implies that the integral curves of V near a regular point behave, up to diffeomorphism, just like the x^1-lines in $\mathbb{R}^n$, so that all of the interesting local behavior is concentrated near the singular points. Of