Math 8061, Homework after Sard's Theorem

You do not have to hand this in

1. Construct three smooth vector fields on S^3 that are linearly independent at each point. (*Hint:* think of S^3 as the group of unit quaternions in $\mathbb{R}^4$, and use the group structure.) Conclude that $TS^3 \cong S^3 \times \mathbb{R}^3$.

2. Let C be a circle smoothly embedded in $\mathbb{R}^4$. The goal of this problem is to use Sard's theorem to show that there exists a 3–dimensional hyperplane H, such that the orthogonal projection $\pi : C \to H$ is an embedding.

Let $v \in S^3$ be a unit vector in $\mathbb{R}^4$. Perpendicular to v is a hyperplane $H_v \cong \mathbb{R}^3$. Let $\pi_v : \mathbb{R}^4 \to H_v$ be the orthogonal projection onto this hyperplane. Then we also obtain a smooth map $\pi_v : C \to H_v$

a) Construct a smooth map $f : X \to S^3$, for some suitably constructed manifold X, such that $v \in S^3$ is a critical value of f precisely when $\pi_v : C \to H_v$ fails to be an immersion.

b) Construct a smooth map $g : Y \to S^3$, for some suitably constructed manifold Y, such that $v \in S^3$ is a critical value of g precisely when $\pi_v : C \to H_v$ fails to be 1–1.

c) Now, use Sard's theorem to conclude that almost every $v \in S^3$ is a critical value of neither f nor g, hence the corresponding projection π_v is an embedding.