Math 8061 Homework 9

Due Wednesday, 12/12/12

1. Suppose M and N are connected, oriented smooth n–manifolds, and $f : M \to N$ is an immersion. Prove that f is orientation–preserving everywhere or orientation–reversing everywhere.

2. Let $T^2 \subset \mathbb{R}^4$ be a torus, parametrized by the map

$$f(\theta, \varphi) = (\cos\theta, \sin\theta, \cos\varphi, \sin\varphi).$$

That is, charts on T^2 are local inverses of f. Then T^2 is oriented by the 2–form $d\theta \wedge d\varphi$.

Compute $\int_{T^2} \omega$, where (w, x, y, z) are the coordinates on $\mathbb{R}^4$, and $\omega = xyz\, dw \wedge dy$.

3. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let ω be a closed 1–form on M. Let C be the unit circle, oriented counterclockwise. Prove that ω is an exact form if and only if $\int_C \omega = 0$.

Hint for the "only if" argument: define a function $f : M \to \mathbb{R}$ by

$$f(p) = \int_\gamma \omega,$$

where γ is an arbitrary piecewise–smooth curve from $(1, 0)$ to p. You will need to argue that this is well–defined, i.e., that the definition does not depend on the choice of γ. This is where Stokes' theorem (and invariance under homotopy) should be of help.