Math 8061 Homework 6

Due Wednesday, 10/31/12

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and let $a \in \mathbb{R}$ be a regular value of f. Earlier in class, we showed that $M = f^{-1}(a)$ is a smooth $(n-1)$ manifold. Now, prove that M is orientable.

2. Compute the flow of each of the following vector fields on $\mathbb{R}^2$:

a) $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

b) $Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

c) $Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

d) $W = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$

3. Let the vector fields X, Y, Z be as in the last problem.

a) Check that $[X, Y] \neq 0$, and that their flows do not commute.

b) Check that $[Y, Z] = 0$, and that their flows commute.

4. Consider the following vector fields on $\mathbb{R}^3$:

$$\begin{aligned} X &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \\ Y &= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \\ X &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \end{aligned}$$

Prove that the flow of $aX + bY + cZ$ is a rotation of $\mathbb{R}^3$ about the line $\mathcal{L}$ through the origin and the point (a, b, c). *Hint: first, prove this for* $(a, b, c) = (1, 0, 0)$. *Then, consider the effect of a linear transformation.*

5. (Ungraded exercise) Let $f : M \to N$ be a diffeomorphism, and let X and Y be vector fields on M. Check that $f_*[X, Y] = [f_*X, f_*Y]$.

You can find a proof of this in many places, including Lie's book. But it's a nice exercise in unwinding the definitions to sort this out for yourself, so I recommend doing that.