Math 8061 Homework 1

Due Wednesday, 9/5/12

1. Let M be an n–dimensional manifold with boundary. Prove that the boundary ∂M is an $(n-1)$ dimensional manifold, without boundary.

2. A topological space X is called *connected* if it cannot be expressed as the disjoint union of two non-empty, open sets. X is called *path-connected* if every $x, y \in X$ are joined by a path: that is, there is a continuous map $f : [0,1] \to X$, such that $f(0) = x$ and $f(1) = y$.

a) Prove that every path-connected metric space is connected. You may assume the classical fact that the interval $[0,1]$ is connected.

b) Prove that a connected manifold is path-connected. Note that the book outlines a proof of this in Proposition 1.8, relying on an (unproved) Lemma A.16 in the Appendix. To do this problem, you would need to either prove Lemma A.16, or argue directly.

3. Let $SL(n, \mathbb{R})$ be the set of $n \times n$ matrices with determinant 1, considered as a subspace of $\mathbb{R}^{n^2}$.

a) Prove that $SL(n, \mathbb{R})$ is a manifold of dimension $n^2 - 1$. *Hint:* Think about the degrees of freedom in choosing the image of a standard basis for $\mathbb{R}^n$.

b) Is $SL(n, \mathbb{R})$ compact?

c) Is $SL(n, \mathbb{R})$ connected?

4. Let $O(3)$ be the set of 3×3 matrices A, such that $\|Av\| = \|v\|$ for every vector $v \in \mathbb{R}^3$.

a) Prove that $O(3)$ is a manifold. What is its dimension?

b) Is $O(3)$ compact?

c) Is $O(3)$ connected?