Math 8061 Homework 8

Due Monday, 12/13/10

1. Suppose M and N are connected, oriented smooth n–manifolds, and $f : M \to N$ is an immersion. Prove that f is orientation–preserving everywhere or orientation–reversing everywhere.

2. Let $T^2 \subset \mathbb{R}^4$ be a torus, parametrized by the map

$$f(\theta, \varphi) = (\cos\theta, \sin\theta, \cos\varphi, \sin\varphi).$$

That is, charts on T^2 are local inverses of f. Then T^2 is oriented by the 2–form $d\theta \wedge d\varphi$.

Compute $\int_{T^2} \omega$, where (w, x, y, z) are the coordinates on $\mathbb{R}^4$, and $\omega = xyz\, dw \wedge dy$.

3. Consider the 1–form $\omega = -y\, dx$ on $\mathbb{R}^2$. Let γ be a piecewise smooth, embedded, closed curve in $\mathbb{R}^2$, oriented counterclockwise.

a) Use Stokes' theorem to show that $\int_\gamma \omega$ is equal to the area enclosed inside γ.

b) Explain how this gives a short solution to problem 3(b) of Homework 6.

4. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let ω be a closed 1–form on M. Let C be the unit circle, oriented counterclockwise. Prove that ω is an exact form if and only if $\int_C \omega = 0$.

Hint for the "only if" argument: define a function $f : M \to \mathbb{R}$ by

$$f(p) = \int_\gamma \omega,$$

where γ is an arbitrary piecewise–smooth curve from $(1, 0)$ to p. You will need to argue that this is well–defined, i.e., that the definition does not depend on the choice of γ.