Math 8061 Homework 6

Due Thursday, 11/11/10

1. Consider the following vector fields on $\mathbb{R}^2$:

$$V = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}, \qquad W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Prove that these vector fields do not commute in two ways:

a) by checking that $[V, W] \neq 0$.

b) by computing their flows, and checking that they do not commute.

2. Let $f : M \to N$ be a diffeomorphism, and let X and Y be vector fields on M. Prove that $f_*[X, Y] = [f_*X, f_*Y]$.

3. Let V and W be the vector fields that span the contact structure on $\mathbb{R}^3$:

$$V = Y = \frac{\partial}{\partial y}, \qquad W = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}.$$

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be an embedded smooth curve in the xy plane, such that $\gamma(1) = \gamma(0)$.

a) Prove that there is a smooth curve $\delta : [0, 1] \to \mathbb{R}^3$, such that $\delta'(t)$ is a linear combination of V and W, and such the projection π_{xy} to the xy plane sends $\delta(t)$ to $\gamma(t)$.

Hint: $\delta(t)$ can be defined by integrating $\delta'(t)$. So, define $\delta'(t)$ so that all the requirements are satisfied.

b) Note that since δ projects to γ, the point $\delta(1)$ must be directly above or below $\delta(0)$. Prove that the vertical distance $|\delta(1) - \delta(0)|$ is equal to the area enclosed by γ.

Hint: First, prove this in the special case where γ is the boundary of a rectangle. Then, use rectangles to approximate an arbitrary smooth curve γ.

This is a special case of Stokes' theorem, and the argument in part (b) mimics the proof.

4. Let $\mathcal{D}$ be the distribution on $\mathbb{R}^3$ spanned by

$$X = \frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y}.$$

a) Find an integrable submanifold of $\mathcal{D}$ passing through the origin.

b) Is $\mathcal{D}$ involutive? Explain your answer in light of part (a).