# Math 8061 Homework 5

Due Thursday, 10/28/10

**1.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and let $a \in \mathbb{R}$ be a regular value of $f$. Earlier in class, we showed that $M = f^{-1}(a)$ is a smooth $(n-1)$ manifold. Now, prove that $M$ is orientable.

**2.** Think of $S^3$ as a subset of $\mathbb{C}^2$:

$$S^3 = \{(w,z) \in \mathbb{C}^2 : |w|^2 + |z|^2 = 1\}.$$

Define an equivalence relation $\sim$ on $\mathbb{C}^2 \setminus \{0\}$, where $(w,z) \sim (\lambda w, \lambda z)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Then

$$S^3/\sim \;\cong\; (\mathbb{C}^2 \setminus \{0\})/\sim \;\cong\; \mathbb{CP}^1 \;\cong\; S^2.$$

Prove that the projection map $\pi : S^3 \to S^2$, where $h(w,z) = [w,z]$, is a fibration with fiber $S^1$. This is called the *Hopf fibration*. There are many beautiful pictures online, but spend some time trying to visualize it before searching online.

**3.** Let $M^n$ be a smooth manifold embedded in $\mathbb{R}^N$. As a result of this embedding, every tangent vector $V \in T_pM$ has a well-defined length $|V|$. Define the *unit tangent bundle* of $M$ as

$$UT(M) \;=\; \{(V,p) : V \in T_p(M), |V| = 1\} \;\subset\; TM.$$

Prove that the projection $\pi : UT(M) \to M$ turns $UT(M)$ into a fiber bundle over $M$, with fiber $S^{n-1}$.

**4.** Compute the flow of each of the following vector fields on $\mathbb{R}^2$:

a) $X = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

b) $Y = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$

c) $Z = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

**5.** Consider the following vector fields on $\mathbb{R}^3$:

$$\begin{aligned} X &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \\ Y &= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \\ X &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \end{aligned}$$

Prove that the flow of $aX + bY + cZ$ is a rotation of $\mathbb{R}^3$ about the line $\mathcal{L}$ through the origin and the point $(a,b,c)$.