Math 8061 Homework 2

Due Thursday, 9/23/10

1. Let $\mathbb{CP}^n$ be the complex projective n–space. That is,

$$\mathbb{CP}^n = \mathbb{C}^{n+1}/\sim, \quad \text{where} \quad (z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n) \text{ for } \lambda \in \mathbb{C}\setminus\{0\}.$$

Prove that $\mathbb{CP}^n$ is a smooth manifold, of real dimension $2n$. *Hint:* see page 7 of Lee's book for a construction of charts on $\mathbb{RP}^n$, and generalize this construction to $\mathbb{CP}^n$. You will need to check that the transition maps between these charts are smooth.

2. Smooth structures on a manifold with boundary can be defined as follows. For an arbitrary set $A \in \mathbb{R}^n$, a function $f : A \to \mathbb{R}^n$ is considered smooth at $p \in A$ if there is a smooth function $\tilde{f} : B_\epsilon(p) \to \mathbb{R}^n$ that agrees with f on $A \cap B_\epsilon(p)$. This gives a natural definition of smooth functions on subsets of $\mathbb{H}^n$, the (closed) upper half-space of $\mathbb{R}^n$. Now, for a manifold M^n, charts to $\mathbb{H}^n$ are called smoothly compatible if the transition maps are smooth, and a smooth atlas is defined accordingly.

a) Let M be a manifold *with* boundary, with smooth atlas $\{(U_i, \varphi_i)\}$. Let N be a manifold *without* boundary, with smooth atlas $\{(V_j, \psi_j)\}$. Show that a smooth structure on $M \times N$ can be given by the atlas $\{(U_i \times V_j,\, \varphi_i \times \psi_j)\}$.

b) Suppose that M and N are both manifolds with boundary. What (if anything) will go wrong if one attempts to construct a smooth structure on $M \times N$ using the method of part (a)?

3. Let M be a smooth manifold. The collection $C^\infty(M)$ of smooth functions $f : M \to \mathbb{R}$ has a natural structure as a vector space over $\mathbb{R}$. (In fact, $C^\infty(M)$ is an algebra, because the product of two smooth functions is a smooth function.) Prove that, if $\dim(M) > 0$, the vector space $C^\infty(M)$ is infinite–dimensional.

4. Let M be a smooth manifold, and let $f : M \to \mathbb{R}$ be a positive, continuous function. Prove that there exists a *smooth* function $g : M \to \mathbb{R}$, such that $0 < g(x) < f(x)$ for all $x \in M$.