Math 8061 Homework 1

Due Thursday, 9/9/10

1. Let M be an n–dimensional manifold with boundary. Prove that the boundary ∂M is an $(n-1)$ dimensional manifold, without boundary.

2. A metric space X is called *connected* if it cannot be expressed as the disjoint union of two non-empty, open sets. X is called *path-connected* if every $x, y \in X$ are joined by a path: that is, there is a continuous map $f : [0,1] \to X$, such that $f(0) = x$ and $f(1) = y$.

a) Prove that every path-connected metric space is connected. (You may assume the classical fact that the interval $[0,1]$ is connected.)

b) Prove that a connected manifold is path-connected.

3. Let $SL(n, \mathbb{R})$ be the set of $n \times n$ matrices with determinant 1, considered as a subspace of $\mathbb{R}^{n^2}$.

a) Prove that $SL(n, \mathbb{R})$ is a manifold of dimension $n^2 - 1$. *Hint:* Think about the degrees of freedom in choosing the image of a standard basis for $\mathbb{R}^n$.

b) Is $SL(n, \mathbb{R})$ compact?

c) Is $SL(n, \mathbb{R})$ connected?

4. Let $O(3)$ be the set of 3×3 matrices A, such that $\|Av\| = \|v\|$ for every vector $v \in \mathbb{R}^3$.

a) Prove that $O(3)$ is a manifold. What is its dimension?

b) Is $O(3)$ compact?

c) Is $O(3)$ connected?