Math 8051 Homework 4

Due Wednesday, 10/11/17

1. Show that the function $f(z) = f(x, y) = x^2 + iy^2$ is differentiable on the line $y = x$, and nowhere else.

2. Let G be a region of the complex plane, and let f be a holomorphic function on G. Let $a \in G$ be a point where $f'(a) \neq 0$. Prove that there exist points of $f(G)$ that approach $f(a)$ from every direction.

Note: Later in the semester, we will prove the Open Mapping Theorem: if f is not constant, $f(G)$ is an open set. In particular, this implies that $f(G)$ contains an open disk around $f(a)$.

3. Let $G = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, and let $\log(z)$ be the principal branch of the logarithm on G. For every $a \in \mathbb{C}$, we can define the *principal branch of the power function* z^a via

$$z^a = \exp(a \log z).$$

a) Check that z^a is holomorphic on G. What is its derivative?

b) Show that for all $z \in G$, the real part of $z^{1/2}$ is always positive.

4. Prove that the stereographic projection defined on page 9 is conformal.

5. Let G be the region contained between the circles $|z| = 1$ and $|z + 1| = 2$.

a) Find a $1 - 1$ conformal map that sends G to a strip between two horizontal lines.

b) Find a $1 - 1$ conformal map that sends G to the open unit disk.