Midterm Exam

Math 8051, Fall 2017

You have 80 minutes. Good luck!

Name: ______________________________

TUID: ______________________________

1. __________ (/20 points)

2. __________ (/24 points)

3. __________ (/20 points)

4. __________ (/24 points)

5. __________ (/12 points)

Total __________ (/100 points)

1. [20 points] State the definitions of the following terms.

(a) compact set

A set K is compact if every open cover of K has a finite subcover.

(b) A function $f : X \to Y$ is uniformly continuous

A function $f: (X, d) \to (Y, \rho)$ is uniformly continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $d(x_1, x_2) < \delta$ then $\rho(f(x_1), f(x_2)) < \varepsilon$.

(c) the Cauchy–Riemann equations for a function $f(z) = f(x, y)$

Write $f(x,y) = u(x,y) + iv(x,y)$, where u, v are real-valued. The Cauchy-Riemann equations say $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$.

(d) the winding number $n(\gamma, a)$

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a} \, dz$$

2. [24 points] **True/False/Explain.** State whether each of the following statements true or false. Then explain your answer in a sentence or two. Provide a counterexample where it is relevant. *This problem does not need complete proofs – don't spend time writing them!*

(a) There is an analytic function $f : \mathbb{C} \to \mathbb{C}$ whose real part is $u(x, y) = x^2 - 2y^2$.

False. A consequence of the Cauchy–Riemann equations is that u would have to be harmonic, i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

But $\frac{\partial^2 u}{\partial x^2} = 2$, $\frac{\partial^2 u}{\partial y^2} = -4$, so u is not harmonic.

(b) There is a non-constant analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $Re\, f(z) > 0$ for all z.

False. Suppose such a function existed. Let g be a Möbius transformation mapping the right-half plane to the unit disk. Then $g \circ f$ is bounded, violating Liouville's Theorem.

(c) Let f be a continuous function on a region G. If f is holomorphic, then f has a primitive.

False. Consider $f(z) = \frac{1}{z}$ on $G = \mathbb{C} \setminus \{0\}$. If f had a primitive, then $\int_\gamma f(z)\,dz = 0$, for γ the unit circle. But $\int_\gamma f(z)\,dz = 2\pi i$.

(This would be true if we assume G is simply connected.)

(d) Let f be a continuous function on a region G. If f has a primitive, then f is holomorphic.

True. If $f(z)$ has a primitive $F(z)$, then $F'(z) = f(z)$ by definition. Thus F is holomorphic. But holomorphic functions are infinitely differentiable, so f is infinitely differentiable also.

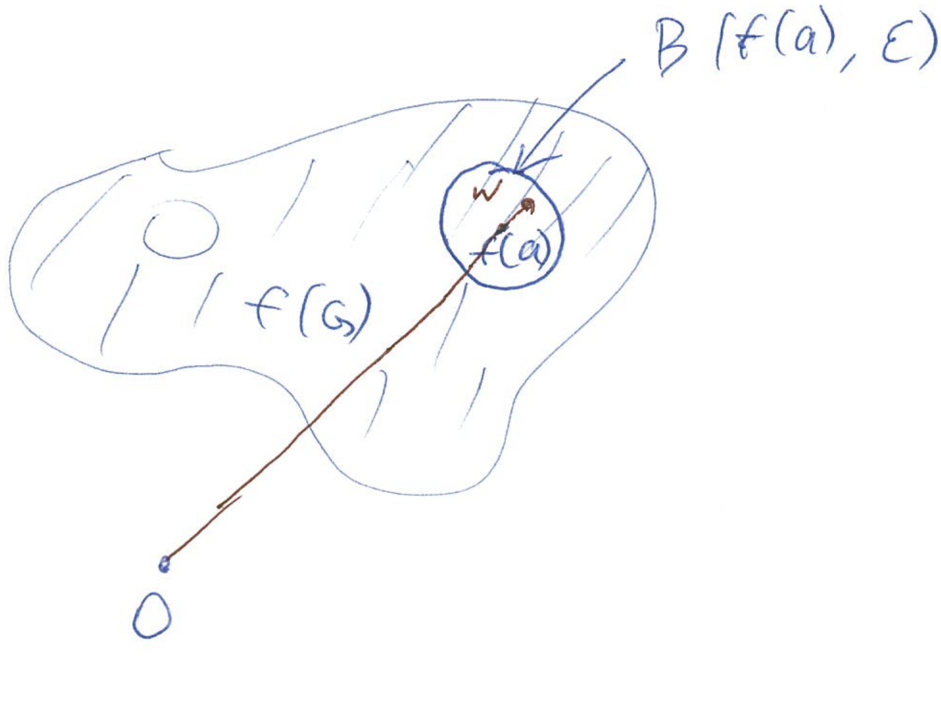
3. [20 points]

(a) State the Maximum Modulus Theorem.

Suppose G is a region, and $f: G \to \mathbb{C}$ is a non-constant analytic function. Then there does not exist $a \in G$ s.t. $|f(a)| \geq |f(z)|$ for all $z \in G$.

(b) The Open Mapping Theorem says that if G is a region, and $f : G \to \mathbb{C}$ is a non-constant analytic function, then $f(G)$ is open in $\mathbb{C}$. Use this statement to prove the Maximum Modulus Theorem.

Let $a \in G$. Then $\exists\, \varepsilon > 0$ s.t. $B(f(a), \varepsilon) \subset f(G)$, by the open mapping theorem. In particular, $\exists\, w \in B(f(a), \varepsilon)$ s.t. $|w| > |f(a)|$. Thus $|f(a)|$ is <u>not</u> the maximum modulus of $|f(z)|$ as z ranges over G.



4. [24 points] Let R be the "wavy ray"

$$R = \{x + iy \in \mathbb{C} : x \leq 0,\ y = \sin(x)\}$$

and let $G = \mathbb{C} \setminus R$.

(a) Show that there is a branch of the logarithm defined on G.

G is simply connected, hence $1/z$ has a primitive on G. A primitive $H(z)$ s.t. $H(1) = 0$ is a branch of the logarithm.

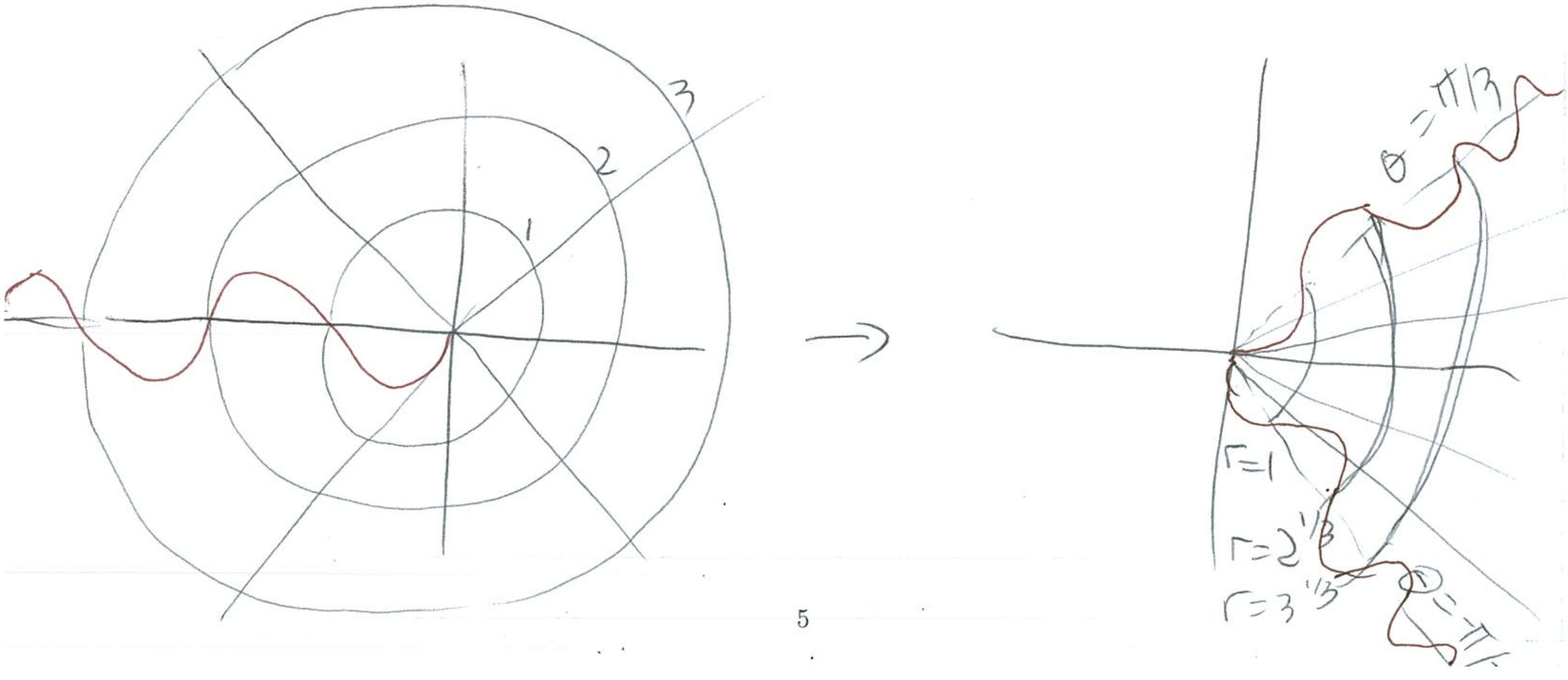
$= z^{1/3}$

(b) Let $f(z) = \exp\left(\frac{1}{3}\log(z)\right)$, where $\log(z)$ is the branch of the logarithm from part (a). Prove that every root of the equation $f(z) = \alpha$ is *simple*, meaning that every zero of $f(z) - \alpha$ has multiplicity 1.

Let $g_\alpha(z) = f(z) - \alpha = z^{1/3} - \alpha$.

Then $g_\alpha'(z) = f'(z) = \frac{1}{3} z^{-2/3} \neq 0$, for any $z \in G$. Since $g_\alpha'(z) \neq 0$, every zero of g_α has multiplicity 1.

(c) Sketch the image $f(G)$, including the images of rays starting at $0 \in \mathbb{C}$ and the images of circles centered at $0 \in \mathbb{C}$.



5. [12 points] Let γ be the circle $\gamma(t) = 4e^{it}$, where $0 \le t \le 2\pi$. Compute

$$\int_\gamma \frac{2z^3}{(z-1)(z-3)}dz.$$

Hint: you can write

$$\frac{2}{(z-1)(z-3)} = \frac{1}{z-3} - \frac{1}{z-1}.$$

If you use any theorems, explain what theorem you are applying, on what region, and check that the hypotheses hold.

Let $f_1(z) = \frac{z^3}{z-3}$, $f_2(z) = \frac{z^3}{z-1}$. Then

$$\int_\gamma \frac{2z^3}{(z-1)(z-3)} dz = \int_\gamma f_1(z) - f_2(z)\, dz.$$

Let $G = \{z : |z| < 5\}$ and $H = \{z : |z| < 4\}$. Then $\gamma = \partial H$, hence $\gamma \approx 0$ in G, i.e. $n(\gamma, a) = 0$ $\forall a \in \mathbb{C} - G$. Thus we may use the Cauchy Integral Formula in G. (Another way to see that γ satisfies the hypotheses is to say γ is null-homotopic in the disk G.) By C. I. F.,

$$\int_\gamma f_1(z)\,dz = \int_\gamma \frac{z^3}{z-3}\,dz = n(\gamma, 3)\cdot(3)^3 = 27 \cdot 2\pi i$$

$$\int_\gamma f_2(z)\,dz = \int_\gamma \frac{z^3}{z-1}\,dz = n(\gamma, 1)\cdot(1)^3 = 1 \cdot 2\pi i$$

Thus $\int_\gamma (f_1 - f_2)\,dz = 26 \cdot 2\pi i = 52\pi i$