Math 4061 Homework 5

Due Tuesday, 3/30/10

1. Let S be the unit sphere in $\mathbb{R}^3$, and let $\sigma : \mathbb{R}^2 \to S$ be the stereographic projection

$$\sigma(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right),$$

which covers all of S except the north pole. Show that the first fundamental form of σ is

$$ds^2 = \frac{4(dx^2+dy^2)}{(1+x^2+y^2)^2}.$$

2. Let S be a *helicoid* (see page 211 for a picture). S can be parametrized by a single chart

$$\sigma(u,v) = (\sinh u \cos v,\ \sinh u \sin v,\ v), \quad \text{where } u > 0.$$

Let T be the *catenoid* obtained by revolving the catenary curve

$$\gamma(u) = (\cosh u,\ 0,\ u)$$

about the z–axis.

(a) Compute the first fundamental form of σ.

(b) Find a parametrization τ of the catenoid T, such that the first fundamental form of τ is the same as that of σ.

(c) Are the surfaces S and T isometric? Are they diffeomorphic?

3. Let $\gamma(t) = (x(t), 0, z(t))$ be a unit–speed curve in the xz plane, which does not intersect the z–axis. Let S be the surface obtained by revolving γ about the z–axis. Prove that

$$\text{area}(S) = 2\pi \int_\gamma x(t)\, dt.$$

(This is called *Pappus's average–radius theorem*, since $\int x(t)\,dt$ divided by the length of γ is the average distance of S from its axis of revolution.)

4. Let $\gamma(u)$ be a unit–speed curve in $\mathbb{R}^3$, whose curvature κ is never 0. Let $N(u)$ and $B(u)$ be the normal and binormal vectors at $\gamma(u)$. Let r be a constant such that $\kappa r < 1$ at every point of γ. The tube of radius r about γ is the surface T parametrized by

$$\sigma(u,v) = \gamma(u) + r\cos(v)\, N(u) + r\sin(v)\, B(u).$$

Prove that $\quad \text{area}(T) = 2\pi r\, \text{length}(\gamma)$.

5. Let T be the torus obtained by revolving the circle $(x-2)^2 + z^2 = 1$ about the z–axis. Compute the area of T using both problem 3 and problem 4, and check that the answers match.