Math 4061 Homework 4

Due Thursday, 2/25/10

1. Let T be the torus in $\mathbb{R}^3$ defined by the equation $(r-2)^2+z^2 = 1$, in cylindrical coordinates. One way to parametrize T is via charts of the form

$$\sigma(\varphi, \theta) = ((2+\cos\varphi)\cos\theta,\ (2+\cos\varphi)\sin\theta,\ \sin\varphi),$$

for coordinates (φ, θ) that each vary in an interval of length less than 2π. One way to picture the dependence on coordinates is the following. If θ is fixed and φ varies, we are walking around a circle of radius 1 in a vertical plane through the z–axis. If φ is fixed and θ varies, we are walking around a circle in a horizontal plane, centered on the z–axis.

(a) How many charts of this type are needed to get an atlas for T?

(c) For given (φ, θ), find a unit normal vector at $\sigma(\varphi, \theta)$. Do these normal vectors depend continuously (and smoothly) on the coordinates?

(b) Prove that T is orientable.

2. Let S be a surface of revolution about a line L. Prove that rotation about L by any angle is a diffeomorphism of S.

3. Let S be a smooth surface in $\mathbb{R}^3$, and let P be a plane that intersects S at the origin and only the origin: $P \cap S = \{0\}$. Prove that $P = T_0S$, the tangent plane to S at 0.

4. Let S be a smooth surface in $\mathbb{R}^3$, and suppose that every normal line to S (that is, every line through $p \in S$ spanned by the normal vector to T_pS) passes through the z–axis.

(a) Let γ be a curve contained in S, whose z–coordinate stays at a constant height h. (In other words, γ is contained in a horizontal plane.) Prove that, for every unit–speed parametrization of γ, the vector $\gamma''(t)$ points toward $(0, 0, h)$.

(b) Prove that the curve γ is part (a) is a circle centered at $(0, 0, h)$.

(c) Prove that S is a surface of revolution about the z–axis.