Midterm Exam

Math 4061, Spring 2010

You have 1 hour and 20 minutes. Good luck!

Name: Solutions

TUID: ______________________

1. ________ (/20 points)

2. ________ (/42 points)

3. ________ (/18 points)

4. ________ (/20 points)

Total ________ (/100 points)

Homework Average ________

Course Average ________

1. [20 points] State the definitions of the following terms.

(a) regular curve

~~[illegible]~~ A curve $\gamma(t)$ is regular if $\gamma'(t) \neq 0$ $\forall t$.

(b) normal vector to a curve in $\mathbb{R}^3$

Given a unit-speed curve $\gamma(t) \in \mathbb{R}^3$, with $\gamma''(t) \neq 0$ (i.e. curvature not zero), the principal normal vector is

$$N(t) = \frac{1}{\kappa} T'(t) = \frac{\gamma''(t)}{|\gamma''(t)|}.$$

(c) smooth surface patch

A surface patch $\sigma(u,v)$ is smooth if all partial derivatives (w.r.t. any combination of u & v) are defined and continuous.

(d) orientable surface

An surface S is orientable if there is an atlas of charts such that all transition maps Φ satisfy $\det(J\Phi) > 0$. Equivalently, S is orientable if there is a continuously varying choice of normal vector at each point of S.

2. [42 points] **True/False/Explain.** State whether each of the following statements is true or false. Then explain your answer, in one or two sentences. Provide a counterexample where it's relevant. *This problem does not need complete proofs – don't spend time writing them!*

(a) The curvature of a regular curve $\gamma(t)$ is the length of the second derivative vector $\gamma''(t)$.

False. This is true for unit-speed curves, but not generally true for regular ones.
For example, let $\gamma(t) = (t^2, 0)$. Then γ is contained in a straight line, so $\kappa = 0$, but $\gamma''(t) \neq 0$.

(b) Let C be the set of all points (x, y) such that $y = \sqrt{x^2 + 10\, e^{\cos x}}$. Then there is a unit-speed parametrization of C.

True. The parametrization $\gamma(t) = (t, \sqrt{t^2 + 10\, e^{\cos t}})$ is regular, so there is a unit-speed reparametrization.

(c) Let $\gamma(t)$ and $\delta(t)$ be curves in $\mathbb{R}^2$, with the property that $\delta(t) = 2\gamma(t)$ for all t. Then the curvature at $\delta(t)$ is twice as big as the curvature at $\gamma(t)$.

False. In fact, the curvature of δ is half that of γ. For example, if γ is a circle of radius 5, its curvature is $1/5$. Then the curvature of δ will be $1/10$.

True/False/Explain, continued.

(d) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a curve whose curvature is everywhere equal to 1 and whose torsion is everywhere equal to 1. Then γ can be moved by rigid motions to be a helix.
Recall: a helix is a spiral of the form $\gamma(t) = (a\cos t, a\sin t, bt)$.

True. A helix has constant curvature and constant torsion. (With the above parametrization, ~~these~~ $\kappa = \frac{|a|}{a^2+b^2}$, $\tau = \frac{b}{a^2+b^2}$, although you don't need to have remembered this.) The point is: there is a helix with curvature 1 and torsion 1, and any other such curve must be equivalent by rigid motions.

(e) The paraboloid $z = x^2 + y^2$ can be parametrized with a single smooth chart.

True. For example, let

$$\sigma(u,v) = (u, v, u^2+v^2).$$

(f) Every surface of revolution is orientable.

True. A curve γ in the plane has a continuously varying normal vector. When γ is revolved to form the surface, the normal vector to γ becomes a normal vector to the surface, varying continuously as we move around the surface.

3. [18 points] Let $\gamma(t) = \left(\frac{4}{5}\cos t,\ 1 - \sin t,\ \frac{-3}{5}\cos t\right)$.

(a) Check that this is a unit-speed parametrization.

$$\gamma'(t) = \left(\frac{-4}{5}\sin t,\ \cos t,\ \frac{3}{5}\sin t\right)$$

$$\text{So } |\gamma'(t)|^2 = \frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t$$
$$= \sin^2 t + \cos^2 t$$
$$= 1.$$

(b) Prove that γ is a circle.

The curvature of a unit-speed curve

$$\text{is } |\gamma''(t)| = \sqrt{\frac{-4}{5}\cos^2 t + \sin^2 t + \frac{9}{25}\cos^2 t}$$
$$= 1.$$

$$\text{So } T = \left(\frac{-4}{5}\sin t,\ \cos t,\ \frac{3}{5}\sin t\right)$$
$$N = \left(\frac{-4}{5}\cos t,\ -\sin t,\ +\frac{3}{5}\cos t\right).$$

$$B = T \times N = \left(\cancel{\tfrac{9}{25}}\ \frac{3}{5}\cos^2 t + \frac{3}{5}\sin^2 t\right)\vec{i}$$
$$+ \left(\frac{12}{25}\sin t\cos t - \frac{12}{25}\sin t\cos t\right)\vec{j}$$
$$+ \left(\frac{4}{5}\sin^2 t + \frac{4}{5}\cos^2 t\right)\vec{k}$$
$$= \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.$$

Thus $B' = 0$, so $\tau = 0$.

Since curvature is constant and torsion is 0, γ is a circle.

4. [20 points] Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, whose gradient vector $\nabla f = (f_x, f_y, f_z)$ is always non-zero. Let S be the surface consisting of all points where $f(x, y, z) = 3$.

(a) Prove that at every point $p = (x, y, z)$ on S, ∇f is perpendicular to the tangent plane T_pS. *Hint:* Consider a smooth curve $\gamma(t)$ on S, passing through p, and prove that $\nabla f \perp \gamma'(t)$.

Write $\gamma(t) = (x(t), y(t), z(t))$. Then, since $\gamma(t)$ lies on S, we have

$$f(x(t), y(t), z(t)) = 3.$$

Differentiating both sides, we get

$$f_x x'(t) + f_y y'(t) + f_z z'(t) = 0$$

$$(f_x, f_y, f_z) \cdot (x'(t), y'(t), z'(t)) = 0.$$

$$\nabla f \cdot \gamma'(t) = 0.$$

Since this is true for every tangent vector γ', ∇f must be perpendicular to the tangent plane T_pS (which consists of tangent vectors $\gamma'(t)$).

(b) Is S orientable? Why or why not?

Yes. We have a vector, namely ∇f, that varies continuously with (x,y,z), and is never 0 and is always perpendicular to S. This means S is orientable.