Review Questions for the Final Exam

Math 320, Fall 2006

1. Are the following true or false? Give a brief explanation or a counterexample.

- If $\sup A \leq \inf B$, and A does not have a maximum, then $a < b$ for all $a \in A$ and $b \in B$.
- If the sequences (a_n) and (b_n) converge, then $(a_n b_n)$ converges.
- Every bounded, monotonic sequence is Cauchy.
- If $\sum a_n$ converges, and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges.
- An open set cannot contain any isolated points.
- If A is a bounded set, then $\sup A$ is a limit point of A.
- Every non-empty compact set contains a non-empty open set.
- If $f : A \to \mathbb{R}$ is differentiable, and $f'(x) > 0$ for all x, then f is 1-to-1.
- If $f : A \to \mathbb{R}$ is differentiable, A is connected, and $f'(x) > 0$ for all x, then f is 1-to-1.
- If f_n converges to f on an interval A, and each f_n is an increasing function, then f is increasing.
- If $f_n \to f$ uniformly on an interval A, and each f_n is differentiable, then f is differentiable.

2. A Buddhist monk leaves his monastery at 7 A.M. and climbs the neighboring mountain, arriving at the top at 7 P.M. After a night of meditation on the mountaintop, he starts descending at 7 A.M. the next day, and arrives at his monastery at 7 P.M. Prove that there is a time t, such that at time t the monk was at the same elevation on both days.

3. Prove that the function $f(x) = \ln x$ is uniformly continuous on $[1, \infty)$. *(Hint: show that $|f'(x)| \leq 1$ on this interval, and use the Mean Value Theorem.)* Is $f(x)$ uniformly continuous on $(0, \infty)$?

4. Let $g(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin(2^n x)}{3^n}$.

Prove that the sum converges on $\mathbb{R}$, and that $g(x)$ is continuous on $\mathbb{R}$. Is g differentiable? Twice differentiable?

5. Let $h(x) = \displaystyle\sum_{n=1}^{\infty} n x^{n-1}$.

Prove that this series converges and defines a continuous function on $(-1, 1)$. *(Hint: what function has $h(x)$ as its derivative?)* Make sure that you reference all necessary theorems in your argument.