Solutions for the Final Exam Review Questions

Math 320, Fall 2006

1. Are the following true or false? Give a brief explanation or a counterexample.

(T) If $\sup A \leq \inf B$, and A does not have a maximum, then $a < b$ for all $a \in A$ and $b \in B$.

(T) If the sequences (a_n) and (b_n) converge, then $(a_n b_n)$ converges.

(T) Every bounded, monotonic sequence is Cauchy.

(F) If $\sum a_n$ converges, and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges.

(T) An open set cannot contain any isolated points.

(F) If A is a bounded set, then $\sup A$ is a limit point of A.

(F) Every non-empty compact set contains a non-empty open set.

(F) If $f : A \to \mathbb{R}$ is differentiable, and $f'(x) > 0$ for all x, then f is 1-to-1.

(T) If $f : A \to \mathbb{R}$ is differentiable, A is connected, and $f'(x) > 0$ for all x, then f is 1-to-1.

(T) If f_n converges to f on an interval A, and each f_n is an increasing function, then f is increasing.

(F) If $f_n \to f$ uniformly on an interval A, and each f_n is differentiable, then f is differentiable.

2. A Buddhist monk leaves his monastery at 7am and climbs the neighboring mountain, arriving at the top at 7pm. After a night of meditation on the mountaintop, he starts descending at 7am the next day, and arrives at his monastery at 7pm. Prove that there is a time t, such that at time t the monk was at the same elevation on both days.

Proof: Let $e_1(t)$ be the monk's elevation at time t on the first day, and $e_2(t)$ be his elevation at time t on the second day. Define a function $f(t) = e_1(t) - e_2(t)$. Then $f(\text{7am}) < 0$, because at 7am the monk was lower on the first day than the second day. Similarly, $f(\text{7pm}) > 0$. Thus, by the Intermediate Value Theorem, there must be a time t such that $f(t) = 0$. At this time, the monk was at the same elevation on both days. □

3. Prove that the function $f(x) = \ln x$ is uniformly continuous on $[1, \infty)$. *(Hint: show that $|f'(x)| \leq 1$ on this interval, and use the Mean Value Theorem.)* Is $f(x)$ uniformly continuous on $(0, \infty)$?

Proof: Since $f'(x) = 1/x$, it follows that $|f'(x)| \leq 1$ on the interval $[1, \infty)$. Now, to prove f is uniformly continuous, let $\epsilon > 0$. We can let $\delta = \epsilon$. For all distinct values x and y in $[1, \infty)$, the Mean Value Theorem says there is a c such that

$$\left|\frac{\ln x - \ln y}{x - y}\right| = \left|\frac{1}{c}\right| \leq 1.$$

Thus, whenever $|x - y| < \delta$, we have

$$|\ln x - \ln y| \leq |x - y| < \delta = \epsilon,$$

implying that $f(x) = \ln x$ is uniformly continuous on $[1, \infty)$.

Meanwhile, f is not uniformly continuous on $(0, \infty)$, because as $x \to 0$, $\ln x \to -\infty$. □

4. Let $g(x) = \sum_{n=1}^{\infty} \frac{\sin(2^n x)}{3^n}$.

Prove that the sum converges on $\mathbb{R}$, and that $g(x)$ is continuous on $\mathbb{R}$. Is g differentiable? Twice differentiable?

Proof: We will prove that the series for $g(x)$ and $g'(x)$ converge uniformly on $\mathbb{R}$. For each $n \in \mathbb{N}$,

$$\left| \frac{\sin(2^n x)}{3^n} \right| \leq \frac{1}{3^n},$$

and $\sum (1/3)^n$ is a convergent geometric series. Thus, by the Weierstrass M–Test, the series for $g(x)$ converges uniformly. Since each function in the series is continuous and the convergence is uniform, $g(x)$ is also continuous.

Similarly, taking the derivative of each function in the series, we get

$$\left| \frac{d}{dx} \frac{\sin(2^n x)}{3^n} \right| = \left| \frac{2^n \cos(2^n x)}{3^n} \right| \leq \frac{2^n}{3^n},$$

and $\sum (2/3)^n$ is a convergent geometric series. Thus the series

$$\sum_{n=1}^{\infty} \frac{2^n \cos(2^n x)}{3^n}$$

converges uniformly to $g'(x)$, and g is differentiable on $\mathbb{R}$. On the other hand, $g''(x)$ is not defined for many values of x, because this time the series is only bounded by $\sum (4/3)^n$. □

5. Let $h(x) = \sum_{n=1}^{\infty} n x^{n-1}$.

Prove that this series converges and defines a continuous function on $(-1, 1)$. *(Hint: what function has $h(x)$ as its derivative?)* Make sure that you reference all necessary theorems in your argument.

Proof: Consider the function

$$f(x) = \sum_{n=0}^{\infty} x^n.$$

We have seen in class that this series converges on the interval $(-1, 1)$. Thus, by the fundamental theorem of power series (Theorem 6.5.7 in the book), $f(x)$ is differentiable on $(-1, 1)$, and its derivative is $f'(x) = h(x)$. Furthermore, again by Theorem 6.5.7, $f(x)$ is differentiable infinitely many times on $(-1, 1)$. Since $f''(x)$ is defined for all $x \in (-1, 1)$, $h(x) = f'(x)$ is differentiable, and therefore continuous. □